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CONTENTS

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Edited by T. W. CHAUNDY, J. HODGKINSON, E. G. C. POOLE
With the co-operation of A. L. DIXON, W. L. FERRAR, G. H. HARDY, A. E. H. LOVE, E. A. MILNE, E. C. TITCHMARSH

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ON A THEOREM CONCERNING IDENTICAL RELATIONS BETWEEN MATRICES

By A. OSTROWSKI (*Basle*)

[Received 23 May 1938]

1. H. B. PHILLIPS gave in 1919† a theorem on identical relations between matrices, which can be stated in the following form:

THEOREM 1. *Let $A_1, A_2,..., A_m$ be a finite number of square matrices of order n and let $x_1,..., x_m$ be m numerical parameters. Let $F(x_1,..., x_m)$ be the determinant of the matrix*

$$x_1 A_1 + ... + x_m A_m. \tag{1.1}$$

Then, if $B_1,..., B_m$ are matrices of order n, commutative with each other and satisfying the equation

$$A_1 B_1 + ... + A_m B_m = 0, \tag{1.2}$$

they will also satisfy the polynomial relation

$$F(B_1,..., B_m) = 0. \tag{1.3}$$

In this theorem it is of course supposed that the polynomial $F(x_1,..., x_m)$ does not vanish identically.

In this case Phillips's result may be improved in the following form:

THEOREM 2. *Under the hypothesis of Theorem* 1, *let*

$$\Phi(x_1,..., x_m)$$

be the greatest common divisor of the n^2 minors of order $n-1$ of the matrix (1.1) *and let*

$$F/\Phi = F^*(x_1,..., x_m).$$

Then the matrices $B_1,..., B_m$ of Theorem 1 *satisfy the equation*

$$F^*(B_1,..., B_m) = 0,$$

and any polynomial $\psi(x_1,..., x_m)$ such that every set of the matrices $B_1,..., B_m$ of Theorem 1 *satisfy the equation*

$$\psi(x_1,..., x_m) = 0$$

contains $F^*(x_1,..., x_m)$ as a factor.

Theorem 1 is a generalization of Cayley's famous theorem: *Every square matrix satisfies its own fundamental equation.* Similarly,

† 'Functions of matrices,' *American J. of Math.* 41 (1919), 267–8.

Theorem 2 corresponds to Frobenius's Theorem about the minimal equation of a square matrix.†

2. My proof of the first part of Theorem 2 proceeds essentially upon the same lines as Phillips's proof of his Theorem 1.

Put $$A_\mu = (a_{ik}^{(\mu)}) \quad (\mu = 1,..., m;\ i,k = 1,..., n)$$

and let $F_{ik}(x_1,..., x_m)$ be the adjoint minors of the determinant

$$F(x_1,..., x_m) = \left|\sum_{\mu=1}^{m} x_\mu a_{ik}^{(\mu)}\right| \quad (i,k = 1,..., n).$$

Then $$\sum_{j=1}^{n}\left(\sum_{\mu=1}^{m} x_\mu a_{jk}^{(\mu)}\right) F_{ji}(x_1,..., x_m) = \delta_i^k F(x_1,..., x_m), \tag{2.1}$$

where as usual $$\delta_i^k = \begin{cases} 0 & (i \lessgtr k), \\ 1 & (i = k). \end{cases}$$

From (2.1) we obtain on dividing by $\Phi(x_1,..., x_m)$

$$\delta_i^k F^*(x_1,..., x_m) = \sum_{j=1}^{n}\left(\sum_{\mu=1}^{m} x_\mu a_{jk}^{(\mu)}\right) F_{ji}^*(x_1,..., x_m), \tag{2.2}$$

where $$F_{ji}^*(x_1,..., x_m) = \frac{F_{ji}(x_1,..., x_m)}{\Phi(x_1,..., x_m)}$$

are polynomials in $x_1,..., x_m$. But now the relations (2.2) are identical relations between polynomials, and therefore remain true if we replace $x_1,..., x_m$ by the matrices $B_1,..., B_m$. Thence we have the relations

$$\delta_i^k F^*(B_1,..., B_m) = \sum_{j=1}^{n} C_{jk} F_{ji}^*(B_1,..., B_m) \quad (i,k = 1,..., n), \tag{2.3}$$

where $$C_{jk} = \sum_{\mu=1}^{m} a_{jk}^{(\mu)} B_\mu. \tag{2.4}$$

On the other hand, if E_{ik} is the matrix of order n with all its elements zero except in the ith row and kth column, we obviously have

$$E_{ij} E_{jk} = E_{ik}, \qquad E_{ij} E_{lk} = 0 \quad (l \gtrless j), \tag{2.5}$$

$$A_\mu = \sum_{i,k=1}^{n} a_{ik}^{(\mu)} E_{ik} \quad (\mu = 1,..., m),$$

and the relation (1.2) can be written using (2.4)

$$\sum_{i,k=1}^{n} E_{ik}\left(\sum_{\mu=1}^{m} a_{ik}^{(\mu)} B_\mu\right) \equiv \sum_{i,k=1}^{n} E_{ik} C_{ik} = 0.$$

† 'Ueber lineare Substitutionen und bilineare Formen', *J. für Math.* 84 (1878), 11–12.

Multiplying this by E_{1j} we have by (2.5)

$$\sum_{k=1}^{n} E_{1k} C_{jk} = 0 \quad (j = 1,..., n). \tag{2.6}$$

But now multiplying (2.6) by $F_{ji}^*(B_1,..., B_m)$ and adding we get

$$0 = \sum_{j=1}^{n} \sum_{k=1}^{n} E_{1k} C_{jk} F_{ji}^* \equiv \sum_{k=1}^{n} E_{1k} \sum_{j=1}^{n} C_{jk} F_{ji}^*(B_1,..., B_m),$$

or, using (2.3),

$$\sum_{k=1}^{n} \delta_i^k E_{1k} F^*(B_1,..., B_m) = 0,$$

$$E_{1i} F^*(B_1,..., B_m) = 0.$$

Here the first row on the left is the ith row of the matrix

$$F^*(B_1,..., B_m) \tag{2.7}$$

and this holds for $i = 1,..., n$. Hence all elements of the matrix (2.7) vanish, which proves the first part of Theorem 2.

3. In the proof of the second part of Theorem 2 it can be assumed without loss of generality that the matrix A_1 is not singular, that is, its determinant does not vanish. Indeed the set (1.1) of matrices contains non-singular matrices, and it is obviously permissible to replace the m matrices A_μ by the m matrices $\sum_{\nu=1}^{m} \alpha_{\mu\nu} A_\nu$ if $|\alpha_{\mu\nu}| \neq 0$. Of course in this case the functions F^* and Φ must be transformed correspondingly.

On the other hand, if A_1 is not singular, the condition (1.2) for the matrices B_μ is equivalent to the condition

$$B_1 + \sum_{\mu=2}^{m} A_1^{-1} A_\mu B_\mu = 0.$$

Hence we can assume from the beginning that A_1 is the unity matrix E, so that

$$A_1 = E.$$

Then the degree of $F(x_1,..., x_m)$ in x_1 is exactly n. We can therefore assume $F^*(x_1,..., x_m)$, a factor of F, in the form

$$F^* = x_1^p + f_1(x_2,..., x_m)x_1^{p-1} + ... + f_p(x_2,..., x_m),$$

where $f_1,..., f_p$ are homogeneous polynomials in $x_2,..., x_m$.

Since $\psi(x_1,..., x_m)$ can be reduced (with modulus F^*) we can assume from the beginning that ψ is a polynomial of degree less than p in x_1. We shall prove that under this assumption ψ must vanish identically.

We now relax the conditions satisfied by $\psi(x_1,\ldots,x_m)$ by specializing the matrices $B_1, B_2,\ldots, B_m$. We put

$$B_2 = -u_2 E, \quad \ldots, \quad B_m = -u_m E,$$

where $u_2,\ldots,u_m$ are numerical parameters. Then it follows from (1.2) that

$$B_1 = u_2 A_2 + \ldots + u_m A_m,$$

and the polynomial ψ is subject to the condition that its degree in x_1 is less than p and that $\psi(u_2 A_2+\ldots+u_m A_m, -u_2,\ldots, -u_m)$ vanishes for all values of $u_2,\ldots, u_m$.

Now for a set of values $u_2,\ldots, u_m$ let the function $f(\lambda) = f(\lambda; u_2,\ldots,u_m)$ be the polynomial of the least degree in λ such that the matrix $\Lambda = u_2 A_2+\ldots+u_m A_m$ satisfies the equation

$$f(\Lambda) = 0.$$

Then obviously $f(\lambda; u_2,\ldots, u_m)$ is a factor of

$$\psi(\lambda; -u_2,\ldots, -u_m). \tag{3.1}$$

If now, for the chosen values $u_2,\ldots, u_m$, $f(\lambda)$ is of degree p, the polynomial (3.1) vanishes identically in λ. It is sufficient for this that for $u_2,\ldots, u_m$ the equation

$$F^*(\lambda; -u_2,\ldots, -u_m) = 0$$

remains the minimum equation for the matrix

$$A^* = u_2 A_2+\ldots+u_m A_m.$$

On the other hand, by the theorem of Frobenius mentioned above, the minimum equation for a matrix A^* is obtained by dividing the determinant of the matrix $\lambda E - A^*$ by the greatest common divisor of the n^2 minors of order $n-1$ of this determinant. If therefore for the assumed set of values $u_2,\ldots, u_m$ the greatest common divisor of the n^2 polynomials

$$F_{ik}(\lambda; -u_2,\ldots, -u_m)$$

is $\Phi(\lambda; -u_2,\ldots, -u_m)$, then (3.1) vanishes identically in λ. But the last condition is equivalent to the condition that the n^2 polynomials

$$F^*_{ik}(\lambda; -u_2,\ldots, -u_m) \quad (i,k = 1,\ldots, n) \tag{3.2}$$

have the greatest common divisor 1 with respect to λ.

Now, since the n^2 polynomials $F^*_{ik}(x_1,\ldots, x_m)$ as polynomials in $x_1,\ldots, x_m$ have the greatest common divisor 1, it is possible to find n^2 polynomials $G_{ik}(x_1,\ldots, x_m)$ such that

$$\sum_{i,k=1}^{n} G_{ik}(x_1,\ldots, x_m) F^*_{ik}(x_1,\ldots, x_m) = H(x_2,\ldots, x_m),$$

where the polynomial H does not vanish identically and is independent of x_1. From this it follows that

$$\sum_{i,k=1}^{n} G_{ik}(\lambda; -u_2,\ldots, -u_m)F^*_{ik}(\lambda; -u_2,\ldots, -u_m) = H(-u_2,\ldots, -u_m).$$

If, therefore, we choose the set $u_2,\ldots, u_m$ such that

$$H(-u_2,\ldots, -u_m) \neq 0,$$

the greatest common divisor of the polynomials (3.2) is 1, and (3.1) vanishes for all values of λ. Hence the product

$$H(-u_2,\ldots, -u_m)\psi(\lambda; -u_2,\ldots, -u_m) \tag{3.3}$$

vanishes for all values of the m variables λ; $u_2,\ldots, u_m$. Thus (3.3) vanishes identically and, since the first factor of (3.3) does not vanish identically, we see that (3.1) and therefore $\psi(x_1,\ldots, x_m)$ vanishes identically which proves the theorem.

ELECTROSTATIC PROBLEMS RELATED TO A PERFORATED STRIP

By A. E. H. LOVE (*Oxford*)

[Received 28 June 1938]

1. By a perforated strip is here meant the plane region bounded externally by two parallel straight lines, and internally by a circle midway between the lines. It is a doubly-connected region. If a functional relation could be found by which this region could be transformed conformally into some standard doubly-connected region, such as that between two concentric circles, the transformation would render possible the solution of some electrical and hydrodynamical problems of considerable interest, and would fill up a well-known gap in the theory of Lamé's equation (**1**). Among the electrical problems in question the two following may be especially noted:

(i) *The Grating Problem.* An infinite set of equidistant parallel straight lines in a plane are the axes of non-intersecting right circular cylinders of equal radii. All the cylinders are the surfaces of conducting bars, of infinite length, which carry equal charges per unit of length. It is required to find the potential at any point.

(ii) *The Condenser Problem.* A conducting right circular cylinder of infinite length has its axis parallel to, and equidistant from, two infinite conducting planes. The planes are at the same potential and the cylinder is at a different potential. It is required to determine the potential at any point.

The condenser problem may be regarded as a case of the grating problem, viz. the case where adjacent bars of the grating carry charges (per unit of length) which are numerically equal, but of opposite signs.

If either of these problems could be solved completely, the solution could be used to effect the conformal transformation of the perforated strip into the standard doubly-connected region, and the solution of the other problem could be deduced.

2. A plane transverse to the axes of the cylinders can be taken to be the plane of a complex variable z ($\equiv x+iy$). Then the potential ϕ is the real part of a function χ of z. We write

$$\chi = \phi+i\psi,$$

and then the curves $\psi =$ constant are lines of force.

In the grating problem the axes of the cylinders are taken to meet the plane in the points given by

$$z = \pm 2nic \quad (n = 0, 1, \ldots),$$

c being real, and the cylinders meet the plane in the circles given by

$$|z \mp 2nic| = a,$$

where $a < c$. These circles are equipotentials, and the lines given by

$$y = \pm(2n+1)c \quad (n = 0, 1, \ldots)$$

are lines of force. The conditions are exactly the same in all the strips bounded by pairs of adjacent lines of this family and perforated by the circles. It is therefore sufficient to consider one such strip, e.g. that between the lines $y = \pm c$, perforated by the circle $|z| = a$. The problem is to be solved by determining χ so that (i) $\phi =$ constant on the circle $|z| = a$, and (ii) $\psi =$ constant on each of the lines $y = \pm c$, the two constant values of ψ being different.

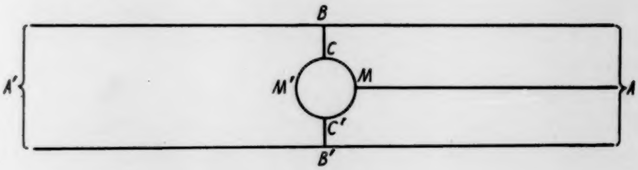

FIG. 1

In the condenser problem the traces of the conducting planes on the z-plane are taken to be the lines $y = \pm c$, and the trace of the conducting cylinder to be the circle $|z| = a$. The problem is to be solved by determining χ so that ϕ has the same constant value on each of the lines $y = \pm c$, and a different constant value on the circle $|z| = a$.

In Fig. 1 the circle $CMC'M'$ is that represented by $|z| = a$, and the lines $A'BA$ and $A'B'A$ are those represented respectively by $y = c$ and $y = -c$. On account of the symmetry of the figure, if either the region $ABCMA$ or the region $ABCMC'B'A$ could be represented conformally upon a half-plane, the conformal transformation of the perforated strip into the standard doubly-connected region could be effected.

In the grating problem points like B and B' where the axis of y intersects the edges of the strip are points of equilibrium. They are

double points on the curves $\phi =$ constant which pass through them, and the lines $\psi =$ constant which pass through them are composite. For example, ψ is the same on the segment BC as it is anywhere on the line $A'BA$. The general appearance of the equipotentials and lines of force would be similar in many respects to those in H. Lamb's drawing of the corresponding curves for a grating of flat strips (**2**).

In the condenser problem the lines BC and $C'B'$ are lines of force.

H. W. Richmond gave an interesting solution of the grating problem (**3**). He showed how the conformal transformation of a region like $ABCMA$ in Fig. 1 to a half-plane could be effected when the circle $CMC'M'$ is replaced by a certain round curve, which differs very little from a circle if the ratio a/c is not too near to 1. This solution does not appear to admit of improvement by any method of successive approximation.

R. C. Knight (**4**) gave a solution of the condenser problem by a method similar to one to be indicated presently for the grating problem. To complete this solution it is necessary to solve an infinite system of linear equations, each containing an infinite number of unknowns. It appears that the system can be solved approximately by numerical computation. This method is of great practical value, but it seemed to me that the labour of carrying it out could be diminished very much by adapting to it the process which I have described elsewhere (**5**) as 'promotion of rank'. By this process the infinite system of equations is replaced by a different system, involving a different set of unknowns, which can be determined one by one. As the grating problem is in some respects simpler than the condenser problem, the solution on the lines indicated above will be developed for the grating problem, and the solution of the condenser problem will be deduced.

3. The solution of the grating problem in the limiting case where $a \to 0$ can be written down in the form (**6**)

$$\chi = -2e_0 \log\{\sinh(\pi z/2c)\}, \tag{3.1}$$

where e_0 is the charge per unit length on a charged line which is at right angles to the plane of (x, y) and passes through the point $z = 0$. In this formula the units are such that the surface density on a charged conductor is the product of $-(1/4\pi)$ and the outward normal derivative of the potential at the surface of the conductor. For example, if lengths are measured in cm., the units can be electro-

static. In general they can be described as 'units of electrostatic type' to distinguish them from the 'rational units' of the types employed by Lorentz and Heaviside.

In what follows we shall simplify the formulae of the problem in the general case by taking

$$e_0 = -\tfrac{1}{2}, \quad a = \lambda\pi, \quad c = \tfrac{1}{2}\pi.$$

Then λ is the ratio $\dfrac{\text{radius of circle}}{\text{breadth of strip}}$.

With these simplifications the function $\log(\sinh z)$ takes the place of the right-hand member of (3.1) as the χ function which gives the solution in the limiting case when $\lambda \to 0$. With this form for χ the values of ψ on the lines $y = \frac{1}{2}\pi$ and $y = -\frac{1}{2}\pi$ are $\frac{1}{2}\pi$ and $-\frac{1}{2}\pi$ respectively.

If a function χ is such that $\mathrm{I}\chi$ is constant along a line, along which y is constant, then $\mathrm{I}(d\chi/dz)$ is zero along that line. Hence from $\log(\sinh z)$ there can be formed by successive differentiation an infinite set of functions whose imaginary parts vanish on the lines $y = \pm\frac{1}{2}\pi$. For the grating problem we shall need only the even differential coefficients. We write

$$w_0 = \log(\sinh z), \qquad w_{2n} = \frac{d^{2n}w_0}{dz^{2n}} \quad (n = 1, 2,\ldots). \tag{3.2}$$

Then we might seek to determine real constants A_2, A_4,... so that

$$\mathrm{R}\Big(w_0 + \sum_{n=1}^{\infty} A_{2n}\, w_{2n}\Big)$$

should be constant on the circle $|z| = \lambda\pi$. If this were done, the solution of the grating problem would be expressed by the formula

$$\chi = w_0 + \sum_{n=1}^{\infty} A_{2n}\, w_{2n}. \tag{3.3}$$

To determine the coefficients A_2, A_4,... we could use the expansion

$$\log(\sinh z) = \log z + \sum_{n=1}^{\infty} (-1)^{n-1}\frac{1}{n} S_{2n}\Big(\frac{z}{\pi}\Big)^{2n}, \tag{3.4}$$

in which $S_{2n} = \sum_{p=1}^{\infty}(p^{-2n})$, and the expansion

$$w_{2n} = -\frac{(2n-1)!}{z^{2n}} + (-1)^{n-1}2(2n-1)!\frac{S_{2n}}{\pi^{2n}} +$$
$$+ \sum_{s=1}^{\infty} (-1)^{n+s-1}2\frac{(2s+2n-1)!}{(2s)!} S_{2n+2s}\frac{z^{2s}}{\pi^{2n+2s}}. \tag{3.5}$$

On introducing polar coordinates r, θ such that $x = r\cos\theta, y = r\sin\theta$, and picking out the coefficient of $\cos 2s\theta$ from the right-hand member of (3.3), it would be found that the solution could be completed by solving for the A_{2n} the system of equations

$$-\frac{(2s-1)!}{(\lambda\pi)^{2s}}A_{2s}+\sum_{n=1}^{\infty}(-1)^{n+s-1}2\frac{(2s+2n-1)!}{(2s)!}S_{2n+2s}\frac{\lambda^{2s}}{\pi^{2n}}A_{2n}+$$

$$+(-1)^{s-1}\frac{1}{s}\lambda^{2s}S_{2s} = 0 \quad (s = 1, 2,\ldots). \qquad (3.6)$$

The constant value of ϕ on $|z| = \lambda\pi$ would then be

$$\log(\lambda\pi)+\sum_{n=1}^{\infty}(-1)^{n-1}2(2n-1)!\,S_{2n}\,\pi^{-2n}A_{2n}.$$

The above would constitute the adaptation of the method of Knight (4) to the grating problem. For the condenser problem the function $\log(\tanh\frac{1}{2}z)$ would take the place of $\log(\sinh z)$.

4. Before proceeding to adapt the idea of 'promotion of rank' we modify the functions w_0, w_{2n} a little by the use of multipliers and added terms, thus writing

$$w_0' = w_0-\log(\lambda\pi), \qquad (4.1)$$

$$w_{2n}' = \frac{(\lambda\pi)^{2n}}{(2n-1)!}w_{2n}-(-1)^{n-1}2\lambda^{2n}S_{2n}. \qquad (4.2)$$

Then on the circle $r = \lambda\pi$ we have trigonometric expansions for $\mathrm{R}w_0'$ and $\mathrm{R}w_{2n}'$ in the forms

$$\mathrm{R}w_0' = \sum_{s=1}^{\infty} a_{0,2s}\cos 2s\theta, \qquad \mathrm{R}w_{2n}' = \sum_{s=1}^{\infty} a_{2n,2s}\cos 2s\theta, \qquad (4.3)$$

where

$$a_{0,2s} = (-1)^{s-1}s^{-1}\lambda^{2s}S_{2s}, \qquad (4.4)$$

$$a_{2n,2s} = (-1)^{n+s-1}2\binom{2n+2s-1}{2s}\lambda^{2n+2s}S_{2n+2s} \quad (n \neq s), \qquad (4.5)$$

$$a_{2n,2n} = -1-2\binom{4n-1}{2n}\lambda^{4n}S_{4n}. \qquad (4.6)$$

As these expansions begin with terms in $\cos 2\theta$ we may describe the functions w_0' and w_{2n}' as being 'of rank 2'.

The process of promotion of rank consists in replacing the functions w_4', $w_6',\ldots$ by functions χ_4, $\chi_6,\ldots$ of ranks equal to their suffixes, that is to say such that the trigonometric expansions of their real parts on the circle begin with terms in $\cos 4\theta$, $\cos 6\theta,\ldots$. For the sake of

completeness we shall write χ_0 and χ_2 in place of w_0' and w_2'. We form χ_4 by the rule

$$\chi_4 = w_4' - q_{4,2}\,\chi_2, \tag{4.7}$$

where $q_{4,2}$ is a constant, adjusted so that the expansion of $R\chi_4$ on the circle may contain no term in $\cos 2\theta$. This condition gives

$$a_{4,2} - q_{4,2}\,a_{2,2} = 0, \tag{4.8}$$

so that it determines $q_{4,2}$. We write the expansion of $R\chi_4$ on the circle

$$R\chi_4 = \sum_{s=2}^{\infty} b_{4,2s}\cos 2s\theta, \tag{4.9}$$

where, for $s \geqslant 2$, $\qquad b_{4,2s} = a_{4,2s} - q_{4,2}\,b_{2,2s}, \qquad$ (4.10)

and, again for the sake of completeness, $b_{2,2s}$ is written for $a_{2,2s}$.

In like manner χ_6, χ_8,... can be formed successively. In general, we write

$$\chi_{2n} = w_{2n}' - \sum_{p=1}^{n-1} q_{2n,2p}\,\chi_{2p}, \tag{4.11}$$

and write the expansion of $R\chi_{2n}$ on the circle as

$$R\chi_{2n} = \sum_{s=n}^{\infty} b_{2n,2s}\cos 2s\theta. \tag{4.12}$$

The coefficients $q_{2n,2p}$ are determined by the set of $n-1$ equations

$$\left.\begin{aligned} a_{2n,2} - q_{2n,2}\,b_{2,2} &= 0 \\ a_{2n,4} - q_{2n,2}\,b_{2,4} - q_{2n,4}\,b_{4,4} &= 0 \\ \cdots\cdots\cdots\cdots & \\ a_{2n,2n-2} - \sum_{p=1}^{n-1} q_{2n,2p}\,b_{2p,2n-2} &= 0 \end{aligned}\right\}, \tag{4.13}$$

and the coefficients $b_{2n,2s}$, for $s \geqslant n$, are given by

$$b_{2n,2s} = a_{2n,2s} - \sum_{p=1}^{n-1} q_{2n,2p}\,b_{2p,2s}. \tag{4.14}$$

5. The solution of the grating problem can now be written down in the form

$$\chi = \chi_0 + \sum_{n=1}^{\infty} B_{2n}\,\chi_{2n} = 0, \tag{5.1}$$

where the coefficients B are determined successively by the equations

$$\left.\begin{aligned} a_{0,2} + B_2\,b_{2,2} &= 0 \\ a_{0,4} + B_2\,b_{2,4} + B_4\,b_{4,4} &= 0 \\ \cdots\cdots\cdots\cdots & \\ a_{0,2n} + \sum_{p=1}^{n} B_{2p}\,b_{2p,2n} &= 0 \\ \cdots\cdots\cdots\cdots & \end{aligned}\right\}. \tag{5.2}$$

6. The function expressed by the right-hand member of (5.1), where the coefficients B_{2n} are determined by (5.2) will be denoted by $F(z,\lambda)$. It has the properties:

(i) $\mathrm{R}F(z,\lambda) = 0$ on the circle $|z| = \lambda\pi$,

(ii) $\mathrm{I}F(z,\lambda) = \pm\frac{1}{2}\pi$ on the lines $\mathrm{I}z = \pm\frac{1}{2}\pi$.

It is made up from the functions w_0, w_2,..., which are independent of λ, numbers, such as S_2, S_4,..., which also are independent of λ, and constants, such as $q_{4,2}$ and B_2, which depend upon λ.

The functions w_{2n} ($n = 1, 2,...$) are polynomials in s^{-1}, where

$$s = \sinh z. \tag{6.1}$$

They are formed successively by the use of the rule

$$\frac{d^2}{dz^2} = \frac{d^2}{ds^2} + \left(s\frac{d}{ds}\right)^2, \tag{6.2}$$

which can be proved easily.

It is found that

$$\left.\begin{aligned}
w_2 &= -\frac{1}{s^2}\\
w_4 &= -(3!)\left(\frac{1}{s^4}+\frac{2^2}{3!}\frac{1}{s^2}\right)\\
w_6 &= -(5!)\left(\frac{1}{s^6}+\frac{1}{s^4}+\frac{2^4}{5!}\frac{1}{s^2}\right)\\
w_8 &= -(7!)\left(\frac{1}{s^8}+\frac{4}{3}\frac{1}{s^6}+\frac{2}{5}\frac{1}{s^4}+\frac{2^6}{7!}\frac{1}{s^2}\right)\\
w_{10} &= -(9!)\left(\frac{1}{s^{10}}+\frac{5}{3}\frac{1}{s^8}+\frac{7}{9}\frac{1}{s^6}+\frac{17}{189}\frac{1}{s^4}+\frac{2^8}{9!}\frac{1}{s^2}\right)\\
w_{12} &= -(11!)\left(\frac{1}{s^{12}}+2\frac{1}{s^{10}}+\frac{19}{15}\frac{1}{s^8}+\frac{256}{945}\frac{1}{s^6}+\frac{62}{4725}\frac{1}{s^4}+\frac{2^{10}}{11!}\frac{1}{s^2}\right)\\
w_{14} &= -(13!)\left(\frac{1}{s^{14}}+\frac{7}{3}\frac{1}{s^{12}}+\frac{28}{15}\frac{1}{s^{10}}+\frac{16}{27}\frac{1}{s^8}+\frac{26}{405}\frac{1}{s^6}+\frac{2}{1485}\frac{1}{s^4}+\frac{2^{12}}{13!}\frac{1}{s^2}\right)
\end{aligned}\right\}. \tag{6.3}$$

In general, if these polynomials are written

$$w_{2n} = -\{(2n-1)!\}\left(A_{n,0}\,s^{-2n}+\sum_{p=1}^{n-1}A_{n,p}\,s^{-2n+2p}\right), \tag{6.4}$$

$A_{n,0} = 1$, $A_{n,n-1} = 2^{2n-2}/(2n-1)!$, and the rule (6.2) gives

$$2n(2n+1)A_{n+1,p} = (2n-2p)(2n-2p+1)A_{n,p}+(2n-2p+2)^2A_{n,p-1}. \tag{6.5}$$

The numbers S_2, S_4,... have been tabulated by Glaisher (**7**).

With any value of λ, not too near to $\frac{1}{2}$, as many of the coefficients such as $q_{4,2}$ and B_2 as may be needed can be computed without much difficulty.

7. The relation
$$\chi = F(z, \lambda) \tag{7.1}$$
effects a conformal representation of the indented half-strip $ABCMC'B'A$ in the plane of z, shown in Fig. 2a, upon the half-strip in the plane of χ, bounded by the lines

$$\psi = \tfrac{1}{2}\pi,\ \phi > 0; \qquad \phi = 0,\ \tfrac{1}{2}\pi > \psi > -\tfrac{1}{2}\pi; \qquad \psi = -\tfrac{1}{2}\pi,\ \phi > 0.$$

This boundary is shown in Fig. 2b, where the same letters as in Fig. 2a are used to indicate corresponding points.

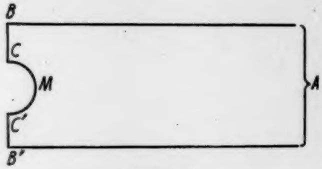

FIG. 2a

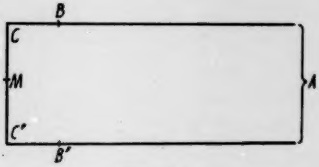

FIG. 2b

The χ region can be represented conformally on the half-plane $\mathrm{I}t > 0$ in the plane of an auxiliary complex variable t by the relation
$$t = i \sinh \chi. \tag{7.2}$$
This has been adjusted so that t is infinite at A; $t = -1$ at C; $t = 0$ at M; $t = 1$ at C'. The values of t at B and B' are respectively $-\cosh \beta$ and $\cosh \beta$, where β is the value of $\mathrm{R}F(z, \lambda)$ at $z = \pm\frac{1}{2}i\pi$. This β is the value of the potential in the grating problem at the points of equilibrium such as B and B'.

In future we shall write $1/k$ for $\cosh \beta$.

The relation
$$t = i \sinh\{F(z, \lambda)\} \tag{7.3}$$
effects a conformal representation of the indented half-strip $ABCMC'B'A$ of Fig. 2a upon the half-plane $\mathrm{I}t > 0$ in such a way that, while $t \to \infty$ at A, its values at B, C, C', B' are $-1/k$, -1, 1, $1/k$.

8. For the condenser problem it is convenient to write
$$W(\equiv U + iV) = i\chi(\equiv -\psi + i\phi), \tag{8.1}$$
so that V is the potential. To solve the problem it is necessary to determine W, as a function of z, so that, in Fig. 2a, V may have a

constant value, which can be taken to be 0, on the semicircle $C'MC$, and may have a constant value, which can be taken to be K', on each of the lines AB, $B'A$, while U has a constant value, which can be taken to be $-K$, on the line BC, and has a constant value, which can be taken to be K, on the line $C'B'$. The solution is expressed by the relation which effects a conformal transformation of a rectangle in the W-plane into the indented half-strip in the z-plane. In Fig. 3, points on the rectangular boundary that correspond to points on the boundary of the indented half-strip are indicated by the same letters as in Fig. 2a.

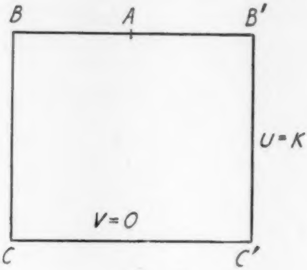

FIG. 3

The W-region is represented conformally on the half-plane $\mathrm{I}t > 0$, with the desired correspondence of particular points, by the relation

$$t = \operatorname{sn} W, \tag{8.2}$$

provided that the periods $4K$ and $2iK'$ of the Jacobian elliptic function answer to a modulus k equal to $\operatorname{sech}\beta$.

Hence the relation

$$\operatorname{sn} W = i \sinh\{F(z, \lambda)\}, \tag{8.3}$$

in which the modulus of the elliptic function is $\operatorname{sech}\beta$, where β is the value of $\mathrm{R}F(z,\lambda)$ at $z = \pm\frac{1}{2}i\pi$, gives the solution of the condenser problem.

The potentials on the conducting surfaces have been chosen to be 0 on the cylinder and K' on the planes. The values of U on the lines BC and $C'B'$ have been taken to be $-K$ and K. In both it is only the difference, not the actual value, which has any physical significance. To give an account of the meaning of the difference between the two values of U we write ds for an element of arc of the semicircle $C'MC$, measured from C' towards C, and $d\nu$ for an

element of the normal to this curve, drawn into the z-region. The charge per unit length on the cylinder is

$$2\int -\frac{1}{4\pi}\frac{\partial V}{\partial \nu}\,ds,$$

where the integral is taken along the curve from C' to C. Now

$$\frac{\partial V}{\partial \nu} = -\frac{\partial U}{\partial s},$$

and therefore this charge is $-K/\pi$. When the potential on the cylinder is less than that on the planes by K', the charge per unit of length on the cylinder is $-K/\pi$. It follows that the capacity per unit of length (C) of the condenser is given by

$$C = K/(\pi K'), \tag{8.4}$$

where the modulus k is $\operatorname{sech}\beta$, and β is the value of $\mathrm{R}F(z,\lambda)$ at $z = \pm\frac{1}{2}i\pi$.

By means of the formula (8.4) I computed the values of C answering to values of λ from 0·05 to 0·45 at intervals of 0·05. For the smaller values of λ it was found to be appropriate to use the known results (**8**)

$$\pi K' = K\log(1/q), \qquad q = \epsilon + 2\epsilon^5 + 15\epsilon^9 + \dots,$$
$$2\epsilon = (1-\sqrt{k'})/(1+\sqrt{k'}),$$

where k' is the complementary modulus, equal to $\tanh\beta$ when k is $\operatorname{sech}\beta$. For the larger values of λ, when k' becomes rather small, it was found to be more convenient to use the approximate formula

$$\frac{K}{\pi K'} = \frac{2}{\pi^2}\left(\log\frac{4}{k'} - \frac{k'^2}{4}\right).$$

These capacities, with the exception of that for $\lambda = 0{\cdot}45$, were computed also by Knight (**4**). He used rational units, and expressed his capacities as values of $2\pi D_0$, where D_0 is a coefficient which he evaluated. The corresponding capacities in electrostatic units would be the values of $\frac{1}{2}D_0$.

The following table gives the values of C, computed by me, and the values of D_0, computed by Knight:

TABLE

λ	0·05	0·1	0·15	0·2	0·25	0·3	0·35	0·4	0·45
C	0·19653	0·27014	0·34606	0·43264	0·53805	0·67492	0·86841	1·1828	1·870
D_0	0·3931	0·5403	0·6921	0·8653	1·0761	1·3498	1·7369	2·3656	

9. It is natural to seek to extend the analysis of § 4 above to the case of a grating placed in a uniform field, parallel to the axis of x, or at right angles to the plane that passes through the axes of the bars, with a view to throwing further light on the theory of thermionic tubes (**9**). As, however, the approximation cannot be a good one unless the distance between the axes of adjacent bars of the grating is small compared with the distance between the (plane) anode and cathode conductors, it seems to be unnecessary to enter into much detail.

We may take the potential due to the uniform field in the absence of the grating to be $x/\lambda\pi$, so that its value on the circle $r = \lambda\pi$ is simply $\cos\theta$. We have to find χ so that, on this circle, $\mathrm{R}\chi+\cos\theta = 0$, and so that, on the lines $y = \pm\frac{1}{2}\pi$, $\mathrm{I}\chi = 0$.

For this we introduce, the infinite set of functions w_{2n+1} given by

$$w_{2n+1} = \frac{d^{2n+1}w_0}{dz^{2n+1}} \quad (n = 0, 1, \ldots). \tag{9.1}$$

From (3.5) we have, on writing $m+1$ for s, the expansion, valid near the circle $|z| = \lambda\pi$,

$$w_{2n+1} = \frac{(2n)!}{z^{2n+1}} + \sum_{m=0}^{\infty} (-1)^{n+m} 2\frac{(2m+2n+1)!}{(2m+1)!} S_{2n+2m+2} \frac{z^{2m+1}}{\pi^{2n+2m+2}}. \tag{9.2}$$

Then we modify this set by means of multipliers, writing

$$w'_{2n+1} = \frac{(\lambda\pi)^{2n+1}}{(2n)!} w_{2n+1}, \tag{9.3}$$

with the expansion, valid near the circle,

$$w'_{2n+1} = \left(\frac{\lambda\pi}{z}\right)^{2n+1} + \sum_{m=0}^{\infty} (-1)^{m+n} 2\binom{2m+2n+1}{2m+1} S_{2n+2m+2} \lambda^{2n+1} \left(\frac{z}{\pi}\right)^{2m+1}. \tag{9.4}$$

For the process of promotion of rank we replace the functions w'_{2n+1}, which are all of rank 1, by a set of functions χ_{2n+1}, of ranks equal to their suffixes, the particular function χ_1 being the same as w'_1. To obtain these functions we write the trigonometric expansion of Rw'_{2n+1} on the circle $r = \lambda\pi$ as

$$\mathrm{R}w'_{2n+1} = \sum_{m=0}^{\infty} a_{2n+1,2m+1} \cos(2m+1)\theta, \tag{9.5}$$

where

$$\left.\begin{aligned} a_{2n+1,2m+1} &= (-1)^{m+n} 2\binom{2n+2m+1}{2m+1} S_{2n+2m+2} \lambda^{2n+2m+2} \quad (m \neq n) \\ a_{2n+1,2n+1} &= 1+2\binom{4n+1}{2n+1} S_{4n+2} \lambda^{4n+2} \end{aligned}\right\}. \tag{9.6}$$

Then we form χ_3, for example, by the rule

$$\chi_3 = w_3' - q_{3,1}\chi_1, \tag{9.7}$$

adjusting $q_{3,1}$ so that the expansion of $\mathrm{R}\chi_3$ on the circle shall have no term in $\cos\theta$. This condition gives

$$a_{3,1} - q_{3,1}a_{1,1} = 0. \tag{9.8}$$

We write the expansion of $\mathrm{R}\chi_3$ on the circle in the form

$$\mathrm{R}\chi_3 = \sum_{m=1}^{\infty} b_{3,2m+1}\cos(2m+1)\theta, \tag{9.9}$$

where, for $m \geqslant 1$,

$$b_{3,2m+1} = a_{3,2m+1} - q_{3,1}b_{1,2m+1}, \tag{9.10}$$

$b_{1,2m+1}$ being written for $a_{1,2m+1}$.

In like manner, in general, we write

$$\chi_{2n+1} = w_{2n+1}' - \sum_{p=0}^{n-1} q_{2n+1,2p+1}\chi_{2p+1}, \tag{9.11}$$

determine the q's by the conditions

$$\left.\begin{aligned} a_{2n+1,1} - q_{2n+1,1}b_{1,1} &= 0 \\ a_{2n+1,3} - q_{2n+1,1}b_{1,3} - q_{2n+1,3}b_{3,3} &= 0 \\ \cdots\cdots\cdots\cdots & \\ a_{2n+1,2n-1} - \sum_{p=0}^{n-1} q_{2n+1,2p+1}b_{2p+1,2n-1} &= 0 \end{aligned}\right\}, \tag{9.12}$$

and write the expansion of $\mathrm{R}\chi_{2n+1}$ on the circle as

$$\mathrm{R}\chi_{2n+1} = \sum_{m=n}^{\infty} b_{2n+1,2m+1}\cos(2m+1)\theta, \tag{9.13}$$

where, for $m \geqslant n$,

$$b_{2n+1,2m+1} = a_{2n+1,2m+1} - \sum_{p=0}^{n-1} q_{2n+1,2p+1}b_{2p+1,2m+1}. \tag{9.14}$$

The solution of the problem of the grating in the uniform field is then of the form

$$\chi = \sum_{n=0}^{\infty} B_{2n+1}\chi_{2n+1}, \tag{9.15}$$

where the constants B are given by the equations

$$\left.\begin{aligned} 1 + B_1 b_{1,1} &= 0 \\ B_1 b_{1,3} + B_3 b_{3,3} &= 0 \\ B_1 b_{1,5} + B_3 b_{3,5} + B_5 b_{5,5} &= 0 \\ \cdots\cdots\cdots\cdots & \end{aligned}\right\}, \tag{9.16}$$

and so they can be found successively.

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ON MINKOWSKI'S THEORY OF REDUCTION OF POSITIVE DEFINITE QUADRATIC FORMS

By K. MAHLER (*Manchester*)

[Received 8 June 1938]

MINKOWSKI* called a positive definite quadratic form in n variables

$$F(x) = \sum_{h,k=1}^{n} a_{hk} x_h x_k$$

reduced, if, for $h = 1, 2,..., n$ and for all systems of n integers $x_1,..., x_n$,

$$F(x) \geqslant a_{hh},$$

when the greatest common divisor

$$\text{g.c.d.}\,(x_h, x_{h+1},..., x_n) = 1,$$

and if certain $n-1$ other unimportant inequalities were satisfied. He proved that, for reduced forms of discriminant D,

$$\lambda_n a_{11} a_{22} \ldots a_{nn} \leqslant D,$$

where $\lambda_n > 0$ depends only on n. L. Bieberbach and I. Schur† showed that

$$\lambda_n \geqslant (\tfrac{48}{125})^{\frac{1}{6}(n^3-n)},$$

and R. Remak‡ in a recent paper improved this to

$$\lambda_n \geqslant \gamma_n (\tfrac{4}{5})^{\frac{1}{2}(n-3)(n-4)}$$

where γ_n is Hermite's constant, for which $D \geqslant \gamma_n a_{nn}^n$.§

As Remak's proof is rather long, I give here a very short and simple proof (which I had obtained before the paper of Remak appeared) for the slightly weaker inequality (since $\frac{4}{9} < \frac{4}{5}$)

$$\lambda_n \geqslant 2^{-2n} (\tfrac{4}{9})^{\frac{1}{2}(n-1)(n-2)} \frac{\{\Gamma(\frac{1}{2})\}^{2n}}{\{\Gamma(1+\frac{1}{2}n)\}^2}.$$

My proof is valid for the reduction of arbitrary convex bodies; it employs Minkowski's theorem on the successive minima of a convex body.||

* *Gesammelte Abhandlungen*, Bd. 2, 53–100.

† *Sitzungsber. Preussische Akad. Wiss., phys.-math. Kl.* (1928), 510–35.

‡ *Compositio Math.* 5 (1938), 368–91.

§ See J. F. Koksma, 'Diophantische Approximationen', *Erg. d. Math.* IV 4, Kap. II, § 6. The best known result for large n is

$$\gamma_n \geqslant \frac{\pi^n}{2^n \Gamma(2+\frac{1}{2}n)^2},$$

due to Blichfeldt.

|| *Geometrie der Zahlen*, 218.

1. Let $f(x) = f(x_1,\dots,x_n)$ be a real function of n real variables $x_1,\dots, x_n$ $(n \geqslant 2)$ with the following properties:

(i) $f(0,\dots,0) = 0, \quad f(x_1,\dots,x_n) > 0$ for $\sum_{h=1}^{n} x_h^2 > 0$;

(ii) $f(tx_1,\dots,tx_n) = |t|f(x_1,\dots,x_n)$ for real t;

(iii) $f(x_1+y_1,\dots,x_n+y_n) \leqslant f(x_1,\dots,x_n)+f(y_1,\dots,y_n)$.

Then, for $t > 0$, the inequality $f(x) \leqslant t$ defines a convex body $K(t)$ in n dimensions, of volume $J(t) = Jt^n$, where J denotes the volume of the body $K(1)$, say K.

For every $t > 0$, since $K(t)$ contains only a finite number of lattice points, it is possible to apply Minkowski's method of reduction* to the function $f(x)$. Let M_h $(h = 1,\dots,n)$ be the set of all lattice points $(x_1,\dots,x_n)$ whose last $n-h+1$ coordinates x_h, $x_{h+1},\dots,x_n$ are relatively prime, and let

$$a_h = f(\delta_{h1}, \delta_{h2},\dots, \delta_{hn}) = f(\delta_h) \quad (h = 1, 2,\dots, n),$$

where δ_{hk} is Kronecker's symbol:

$$\delta_{hh} = 1, \quad \text{but} \quad \delta_{hk} = 0 \text{ for } h \neq k \quad (h, k = 1, 2,\dots, n).$$

Definition. *The function $f(x)$ and the corresponding convex body K are called 'reduced', if for each $h = 1,\dots, n$ and for all lattice points (x) in M_h*

$$f(x) \geqslant a_h.$$

As in Minkowski's paper, it is easily proved that $f(x)$ can be reduced by applying a suitable unimodular linear transformation

$$x_h \to \sum_{k=1}^{n} a_{hk} x_k \quad (h = 1, 2,\dots, n)$$

with integer coefficients.

2. Theorem. *For reduced functions $f(x)$*

$$a_1 a_2 \dots a_n \leqslant \frac{2^n (\frac{3}{2})^{\frac{1}{2}(n-1)(n-2)}}{J}.$$

Proof. Minkowski† proved that there are n independent lattice points

$$(p_h) = (p_{h1},\dots, p_{hn}) \quad (h = 1, 2,\dots, n)$$

such that, if

$$S_h = f(p_h) = f(p_{h1},\dots, p_{hn}) \quad (h = 1, 2,\dots, n),$$

then

$$S_1 \leqslant S_2 \leqslant \dots \leqslant S_n, \qquad S_1 S_2 \dots S_n \leqslant \frac{2^n}{J},$$

* *Gesammelte Abhandlungen*, Bd. 2, 53–100.

† *Geometrie der Zahlen*, 218.

and $$f(x) \geqslant S_h$$
for all lattice points (x) which are linearly independent of (p_1), $(p_2),\ldots, (p_{h-1})$.

Obviously, (p_1) belongs to M_1; hence
$$a_1 \leqslant S_1. \tag{1}$$
(More exactly $a_1 = S_1$, but we do not need this.)

Suppose that we have already obtained $m-1$ positive absolute constants
$$\gamma_1, \gamma_2,\ldots, \gamma_{m-1},$$
such that
$$a_h \leqslant \gamma_h S_h \quad (h = 1, 2,\ldots, m-1). \tag{2}$$
By (1) in particular $\gamma_1 = 1$,

and we now find a similar constant γ_m for which
$$a_m \leqslant \gamma_m S_m. \tag{3}$$

The m lattice points $(p_1),\ldots, (p_m)$ are independent. Hence at least one of them, say $(p_i) = (p_{i1},\ldots, p_{im})$ ($i = 1$ or 2 or...or m), has its last $n-m+1$ coordinates $p_{im}, p_{i,m+1},\ldots, p_{in}$ not all zero. Therefore the greatest common divisor
$$\text{g.c.d. } (p_{im}, p_{i,m+1},\ldots, p_{in}) = d_m \neq 0.$$
If $d_m = 1$, then (p_i) belongs to M_m, and therefore $a_m \leqslant S_i$,

i.e.
$$a_m \leqslant S_m. \tag{4}$$
Suppose, however, that $d_m \geqslant 2$. Then we can find $m-1$ integers $g_1, g_2,\ldots, g_{m-1}$, such that
$$p_{ih}+g_h \equiv 0 \pmod{d_m} \quad \text{and} \quad |g_h| \leqslant \tfrac{1}{2}d_m \quad (h = 1, 2,\ldots, m-1).$$
Hence, writing the left-hand side in vector form,
$$\left(\frac{p_{i1}+g_1}{d_m},\ldots,\frac{p_{i,m-1}+g_{m-1}}{d_m}, \frac{p_{im}}{d_m},\ldots,\frac{p_{in}}{d_m}\right) = \frac{1}{d_m}\left\{\sum_{k=1}^{m-1} g_h(\delta_h)+(p_i)\right\}$$
is a lattice point of the set M_m. Therefore, from (ii) and (iii), since $d_m \geqslant 2$,
$$a_m \leqslant \frac{1}{d_m}\left\{\sum_{h=1}^{m-1} |g_h|a_h+S_i\right\} \leqslant \frac{1}{2}\left\{\sum_{h=1}^{m-1} \gamma_h S_h+S_m\right\},$$
i.e.
$$a_m \leqslant \frac{\gamma_1+\gamma_2+\ldots+\gamma_{m-1}+1}{2} S_m. \tag{5}$$

Put
$$\gamma_m = \max\left(1, \frac{\gamma_1+\gamma_2+\ldots+\gamma_{m-1}+1}{2}\right).$$
Then, from (4), (5), we see that (3) is satisfied.

Now

$$\gamma_1 = 1, \qquad \gamma_2 = \max\left(1, \frac{1+1}{2}\right) = 1,$$

$$\gamma_3 = \max\left(1, \frac{1+1+1}{2}\right) = \tfrac{3}{2},$$

$$\gamma_4 = \max\left(1, \frac{1+1+\frac{3}{2}+1}{2}\right) = \tfrac{9}{4} = (\tfrac{3}{2})^2.$$

Suppose, then, that

$$\gamma_h = (\tfrac{3}{2})^{h-2} \quad \text{for} \quad h = 2, 3, \ldots, m-1. \tag{6}$$

Then

$$\gamma_m = \max\left(1, \frac{3+(\frac{3}{2})^1+(\frac{3}{2})^2+\ldots+(\frac{3}{2})^{m-3}}{2}\right) = \frac{1}{2}\left\{3+\frac{(\frac{3}{2})^{m-2}-(\frac{3}{2})^1}{\frac{3}{2}-1}\right\} = (\tfrac{3}{2})^{m-2},$$

and so (6) holds for $h = m$.

On multiplying the inequalities

$$a_1 \leqslant S_1 \quad \text{and} \quad a_h \leqslant (\tfrac{3}{2})^{h-2} S_h \quad \text{for} \quad h = 2, 3, \ldots, n,$$

we have

$$a_1 a_2 \ldots a_n \leqslant (\tfrac{3}{2})^{1+2+\ldots+(n-2)} S_1 S_2 \ldots S_n \leqslant \frac{2^n (\frac{3}{2})^{\frac{1}{2}(n-1)(n-2)}}{J},$$

as was to be proved.

Suppose in particular that

$$\{f(x)\}^2 = F(x) = \sum_{h,k=1}^{n} a_{hk} x_h x_k$$

is a reduced positive definite quadratic form of determinant D. Then

$$J = \frac{\{\Gamma(\frac{1}{2})\}^n}{\Gamma(1+\frac{1}{2}n)} \frac{1}{\sqrt{D}}$$

is the volume of the convex body $f(x) \leqslant 1$, and so by our theorem

$$a_{11} a_{22} \ldots a_{nn} \leqslant 2^{2n} (\tfrac{3}{2})^{(n-1)(n-2)} \frac{\{\Gamma(1+\frac{1}{2}n)\}^2}{\{\Gamma(\frac{1}{2})\}^{2n}} D,$$

since

$$a_{hh} = a_h^2 \quad (h = 1, 2, \ldots, n).$$

AN INTEGRAL EQUATION

By J. L. B. COOPER (*Oxford*)

[Received 16 June 1938]

1. The set of linear equations in an infinite number of unknowns

$$\sum_{n=1}^{\infty} a_n x_{n+m} = 0 \quad (m = 1, 2, \ldots)$$

has been investigated by Titchmarsh.* The investigation of the integral equation which is its analogue

$$\int_x^{\infty} k(y-x) f(y)\, dy = 0 \tag{A}$$

was suggested to me by him and forms the subject of this paper.

The obvious solutions of this equation are exponentials. If

$$K(w) = \frac{1}{\sqrt{(2\pi)}} \int_0^{\infty} k(t) e^{iwt}\, dt$$

vanishes for $w = \alpha$, then $f(y) = e^{i\alpha y}$ is a solution of (A). If $w = \alpha$ is an n-fold zero, $f(y) = P(y)e^{i\alpha y}$ is a solution if $P(y)$ is any polynomial of degree $(n-1)$.

I shall first show that, if $k(t)$ and $f(t)$ satisfy certain conditions of size at infinity, all solutions are of this type. This is proved by the use of complex Fourier transforms.† A theorem of Wiener is then used to prove a result for different order conditions on the functions. In the last section certain analogues of Titchmarsh's results for linear equations are proved; the case where $k(t)$ is monotonically decreasing is discussed, and only ordinary real-variable methods are used.

2. THEOREM 1. *If $e^{at}k(t)$, $e^{-bt}f(t)$ are $L^2(0,\infty)$, for some $a > b > 0$, and $e^{ct}f(t)$ is $L^2(-\infty, 0)$, for some real c, then, if* (A) *is satisfied,*

$$f(y) = \sum R(y) e^{i\alpha y},$$

where α is an n-fold zero of $K(w)$, $R(y)$ is any polynomial of degree $(n-1)$, and the sum is taken over all zeros of $K(w)$ such that

$$-c < I(\alpha) < b.$$

* *Proc. Cambridge Phil. Soc.* 22 (3) 1924, 282–6.

† Cf. Titchmarsh, *Theory of Fourier Integrals*, § 11.2. The methods there used would require severer restrictions on $f(t)$ as $t \to -\infty$ than are needed in Theorem 1.

Write

$$p(x) = \int_0^\infty k(t)f(t+x)\,dt, \qquad q(x) = \int_0^\infty k(t+x)f(t)\,dt.$$

Then

$$\begin{aligned} p(x) &= e^{bx}\int_0^\infty e^{bt}k(t)e^{-b(t+x)}f(t+x)\,dt \\ &\leqslant e^{bx}\left(\int_0^\infty \{e^{bt}k(t)\}^2\,dt\right)^{\frac{1}{2}}\left(\int_0^\infty \{e^{-b(t+x)}f(t+x)\}^2\,dt\right)^{\frac{1}{2}} \\ &= o(e^{bx}), \quad \text{as} \quad x \to \infty. \end{aligned}$$

Similarly, as $x \to \infty$, $q(x) = o(e^{-ax})$.

Again

$$\begin{aligned} p(-x) &= \int_0^\infty k(t)f(t-x)\,dt \\ &= e^{-vx}\int_0^\infty e^{vt}k(t)e^{-v(t-x)}f(t-x)\,dt \quad (a \leqslant v \leqslant b) \\ &\leqslant e^{-vx}\left(\int_0^\infty \{e^{vt}k(t)\}^2\,dt\right)^{\frac{1}{2}}\left(\int_{-x}^\infty \{e^{-vt}f(t)\}^2\,dt\right)^{\frac{1}{2}} \\ &= O\left(e^{-vx}\left[\int_{-x}^0 \{e^{-vt}f(t)\}^2\,dt\right]^{\frac{1}{2}}\right) \\ &= e^{-vx}O\left(\left[\int_{-x}^0 e^{-2(c+v)t}\{e^{ct}f(t)\}^2\,dt\right]^{\frac{1}{2}}\right) \\ &= O(e^{cx}), \quad \text{as} \quad x \to \infty. \end{aligned}$$

Define

$$F_+(w) = \frac{1}{\sqrt{(2\pi)}}\int_0^\infty f(x)e^{iwx}\,dx, \qquad F_-(w) = \frac{1}{\sqrt{(2\pi)}}\int_{-\infty}^0 f(x)e^{iwx}\,dx,$$

and, similarly, $P_+(w)$, $Q_+(w)$, $P_-(w)$, where w is the complex variable $u+iv$.

Because of the order results on the functions involved,

$F_+(w)$ and $P_+(w)$ are analytic for $v > b$,

$F_-(w)$ and $P_-(w)$ are analytic for $v < -c$,

$Q_+(w)$ and $K(w)$ are analytic for $v > -a$.

In what follows, all the integrals are absolutely convergent for the values of v chosen, and so the inversions of the orders of integration are justified.

If $v < a$,

$$\sqrt{(2\pi)}Q_+(-w) = \int_0^\infty e^{-iwx}\,dx \int_x^\infty f(t-x)k(t)\,dt$$

$$= \int_0^\infty k(t)e^{-iwt}\,dt \int_0^t e^{iwx}f(x)dx.$$

If $b < v < a$,

$$\sqrt{(2\pi)}P_+(w) = \int_0^\infty e^{iwx}\,dx \int_0^\infty k(t)f(t+x)\,dt$$

$$= \int_0^\infty e^{-iwt}k(t)\,dt \int_0^\infty e^{iw(t+x)}f(t+x)\,dx$$

$$= \int_0^\infty e^{-iwt}k(t)\,dt \int_0^\infty e^{iwx}f(x)\,dx -$$

$$- \int_0^\infty e^{-iwt}k(t)\,dt \int_0^t e^{iwx}f(x)\,dx$$

$$= 2\pi K(-w)F_+(w) - \sqrt{(2\pi)}Q_+(-w),$$

in view of the result on $Q_+(-w)$ just proved. Hence

$$\sqrt{(2\pi)}K(-w)F_+(w) - P_+(w) = Q_+(-w) \quad (a > v > b),$$

and similarly,

$$\sqrt{(2\pi)}K(-w)F_-(w) - P_-(w) = -Q_+(-w) \quad (v < -c).$$

If (A) is satisfied, $p(x) = 0$ for all x, and so

$$F_+(w) = \frac{1}{\sqrt{(2\pi)}}\frac{Q_+(-w)}{K(-w)} \quad (a > v > b),$$

$$F_-(w) = -\frac{1}{\sqrt{(2\pi)}}\frac{Q_+(-w)}{K(-w)} \quad (v < -c).$$

It follows that $F_+(w)$ and $F_-(w)$ are analytic for $v < a$ and regular save for poles at the zeros of $K(-w)$, and that

$$F_+(w) + F_-(w) = 0.$$

Let v_1, v_2 be such that $a < v_1 < b$, $v_2 < -c$, and $K(-w)$ has no zeros for $v = v_1$ or v_2. Then

$$f(x) = \frac{1}{\sqrt{(2\pi)}}\lim_{\lambda\to\infty}\left\{\int_{iv_1-\lambda}^{iv_1+\lambda} F_+(w)e^{-ixw}\,dw + \int_{iv_2-\lambda}^{iv_2+\lambda} F_-(w)e^{-ixw}\,dw\right\}.$$

Since $F_+(w)$, $F_-(w)$ tend to zero as $u \to \pm\infty$, we can use the calculus of residues to evaluate these integrals: and so we see that $f(x)$ is equal

to the sum of the residues at the poles of $F_+(w)$. This gives the theorem.

3. Methods different to those used above are required to deal with the equation when the difference between the orders of $k(t)$ and $f(t)$ is not as large as we have assumed. Bochner* has treated a similar equation by means of his generalized Fourier integrals, and his results can be applied with very little modification to our equation to prove that, if, as $t \to \pm\infty$, $k(t)$ is $O(t^{-n})$ and $f(t)$ is $O(t^{n-2})$, and if $K(w)$ has only a finite number of real zeros, then $f(t)$ is the sum of exponentials. The following two theorems seem worth mentioning here, both because of their symmetry and because the second imposes even weaker conditions on $k(t)$ and $f(t)$ than are needed by Bochner. We state them for an equation more general than (A) and including (A).

THEOREM 2. *If $k(t)$ is $L(-\infty, \infty)$, then a necessary condition that the equation*

$$\int_{-\infty}^{\infty} k(x-y)f(y)\,dy = g(x)$$

have a solution for every $g(x)$ of $L(-\infty, \infty)$ is that $K(w)$ have no real zeros.

If $K(w)$ has no real zeros, then for every $g(x)$ of $L(-\infty, \infty)$ we can find a sequence of functions $f_n(y)$ of $L(-\infty, \infty)$, such that, if

$$\int_{-\infty}^{\infty} k(x-y)f_n(y)\,dy = g_n(x),$$

then
$$\lim_{n\to\infty} \int_{-\infty}^{\infty} |g(y)-g_n(y)|\,dy = 0.$$

The first part is trivial, for, if

$$\int_{-\infty}^{\infty} k(x-y)f(y)\,dy = g(x),$$

then
$$\sqrt{(2\pi)}K(w)F(w) = G(w)$$

for all real w, and hence the equation can have no solution for those functions $g(x)$ whose transforms do not have the same zeros as $K(w)$.

The second part is contained in Theorem 8 of Wiener's *The Fourier Integral*.

* *Vorlesungen über Fouriersche Integrale*, § 34.

THEOREM 3. *If $k(y)$ is $L(-\infty,\infty)$ and $K(w)$ has no real zeros, there is no bounded solution of*

$$\int_{-\infty}^{\infty} k(y-x)f(y)\,dy = 0$$

other than $f(y) \equiv 0$.

Suppose that $f(y)$ is a bounded solution of the equation, and let $g(x)$ be any function of $L(-\infty,\infty)$. Let $f_n(y)$ be a sequence of functions satisfying the conditions of the last theorem. Then

$$\begin{aligned}\int_{-\infty}^{\infty} f(y)g_n(y)\,dy &= \int_{-\infty}^{\infty} f(y)\,dy \int_{-\infty}^{\infty} k(y-x)f_n(x)\,dx \\ &= \int_{-\infty}^{\infty} f_n(x)\,dx \int_{-\infty}^{\infty} k(y-x)f(y)\,dy \\ &= 0,\end{aligned}$$

inversion being justified by absolute convergence. Now

$$\begin{aligned}\left|\int_{-\infty}^{\infty} f(y)g(y)\,dy\right| &\leqslant \left|\int_{-\infty}^{\infty} f(y)g_n(y)\,dy\right| + \left|\int_{-\infty}^{\infty} f(y)\{g(y)-g_n(y)\}\,dy\right| \\ &= \left|\int_{-\infty}^{\infty} f(y)\{g(y)-g_n(y)\}\,dy\right| \\ &\leqslant \max|f(y)| \int_{-\infty}^{\infty} |g(y)-g_n(y)|\,dy \\ &\to 0 \quad \text{as} \quad n \to \infty.\end{aligned}$$

Hence
$$\int_{-\infty}^{\infty} f(y)g(y)\,dy = 0.$$

This is true for every $g(y)$ of $L(-\infty,\infty)$, and so

$$f(y) \equiv 0.$$

4. The theorems of this section are concerned with the behaviour of $f(y)$ as $y \to \infty$ on the assumption that $k(x)$ is non-increasing and positive. They differ from those above in that for their proof it is not necessary to assume that (A) is satisfied for all x; it is sufficient for it to be satisfied for all x after a certain value.

THEOREM 4. *If $k(t)$ is a non-increasing function of t, and $k(t) > 0$ for all $t > 0$, then* (A) *has no solution $f(t)$, not identically zero, such that $f(t) \to 0$ as $t \to \infty$.*

Let $f(t)$ be a solution of (A). Then, for any two positive numbers x_1, x_2,

$$0 = \int_{x_1}^{\infty} k(y-x_1)f(y)\,dy - \int_{x_2}^{\infty} k(y-x_2)f(y)\,dy$$

$$= \int_{x_1}^{x_2} k(y-x_1)f(y)\,dy - \int_{x_2}^{\infty} f(y)\{k(y-x_2)-k(y-x_1)\}\,dy,$$

and so

$$\int_{x_1}^{x_2} k(y-x_1)f(y)\,dy = \int_{x_2}^{\infty} \{k(y-x_2)-k(y-x_1)\}f(y)\,dy. \tag{1}$$

Now we prove that, if x_2 is in the Lebesgue set of $f(y)$, then, as $x_1 \to x_2$,

$$\int_{x_1}^{x_2} k(y-x_1)f(y)\,dy = (x_2-x_1)[k(0)f(x_2)+o(1)], \tag{2}$$

where we write $k(0)$ for $k(0+)$.

$$\left|\int_{x_1}^{x_2} k(y-x_1)f(y)\,dy - (x_2-x_1)k(0)f(x_2)\right|$$

$$\leqslant \int_{x_1}^{x_2} |f(y)k(y-x_1)-f(x_2)k(x_2-x_1)|\,dy +$$

$$+(x_2-x_1)|f(x_2)|\,|k(x_2-x_1)-k(0)|.$$

The second term is certainly $o(x_2-x_1)$. As for the first,

$$\int_{x_1}^{x_2} |f(y)k(y-x_1)-f(x_2)k(x_2-x_1)|\,dy$$

$$\leqslant \int_{x_1}^{x_2} |k(y-x_1)\{f(y)-f(x_2)\}|\,dy$$

$$+\int_{x_1}^{x_2} |f(x_2)|\,|k(y-x_1)-k(y-x_2)|\,dy$$

$$\leqslant k(0)\int_{x_1}^{x_2} |f(y)-f(x_2)|\,dy + o(x_2-x_1) = o(x_2-x_1),$$

since x_2 is in the Lebesgue set of $f(y)$.

We next define $\mu(x)$ to be the essential upper bound of $|f(y)|$ for $y > x$. By definition of $\mu(x)$ the set of points $y > x$ such that

$$|f(y)| > \rho\mu(x)$$

is of positive measure if $\rho < 1$, of zero measure if $\rho > 1$. If $\rho < 1$, any such set will contain points of the Lebesgue set of $f(y)$.

Now take $x_2 > x$ to be a point in the Lebesgue set of $f(x)$ such that

$$\mu(x_2),\ |f(x_2)| > \rho\mu(x), \tag{3}$$

where ρ is a number between $\frac{3}{5}$ and 1 which will be specified later. From (1)

$$\left|\int_{x_1}^{x_2} f(y)k(y-x_1)\,dy\right| \leqslant \mu(x_2)\int_{x_2}^{X}\{k(y-x_2)-k(y-x_1)\}\,dy+$$
$$+\mu(X)\int_{X}^{\infty}\{k(y-x_2)-k(y-x_1)\}\,dy,$$

for any $X > x_2$. Again

$$\int_{X}^{\infty}\{k(y-x_2)-k(y-x_1)\}\,dy = \lim_{T\to\infty}\int_{X}^{T}\{k(y-x_2)-k(y-x_1)\}\,dy$$
$$= \int_{X-x_2}^{X-x_1} k(t)\,dt - k(\infty)(x_2-x_1),$$

so

$$\left|\int_{x_1}^{x_2} k(y-x_1)f(y)\,dy\right| \leqslant \mu(x_2)\left\{\int_{0}^{x_2-x_1} k(y)\,dy - \int_{X-x_2}^{X-x_1} k(y)\,dy\right\}+$$
$$+\mu(X)\left\{\int_{X-x_2}^{X-x_1} k(y)\,dy - k(\infty)(x_2-x_1)\right\}.$$

If $f(x)\to 0$, $\mu(X)\to 0$ as $X\to\infty$, and so we can find X such that $\mu(X) < \frac{1}{3}\mu(x)$. Then $\mu(X) < \frac{1}{2}\mu(x_2)$ by (3); and

$$\left|\int_{x_1}^{x_2} k(y-x_1)f(y)\,dy\right| \leqslant \mu(x_2)\left\{\int_{0}^{x_2-x_1} k(y)\,dy - \tfrac{1}{2}\int_{X-x_2}^{X-x_1} k(y)\,dy\right\}$$
$$\leqslant \mu(x_2)(x_2-x_1)\{k(0)-\tfrac{1}{2}k(X)\},$$

and so, from (2),

$$(x_2-x_1)\{k(0)|f(x_2)|+o(1)\} \leqslant \mu(x_2)(x_2-x_1)\{k(0)-\tfrac{1}{2}k(X)\};$$

and, making $x_1\to x_2$, we see that

$$k(0)|f(x_2)| \leqslant \{k(0)-\tfrac{1}{2}k(X)\}\mu(x_2).$$

We have $|f(x_2)| > \rho\mu(x)$, and so, if

$$\rho > \{k(0)-\tfrac{1}{2}k(X)\}/k(0),$$
$$\mu(x_2) > \mu(x),$$

and this is a contradiction. So the result follows by *reductio ad absurdum.*

THEOREM 5.* *If $k(t)$ is non-increasing and positive in* $(0,\infty)$, *there is no solution of* (A) *not equivalent to zero such that the integral* $\int^{\infty} f(y)\,dy$ *exists.*

* This theorem and the first of the proofs are due to Professor Titchmarsh.

If there is such a solution, put

$$f_1(x) = -\int_x^\infty f(y)\,dy,$$

so that $f_1(x) \to 0$ as $x \to \infty$. Then

$$\int_x^\infty k(y-x)f(y)\,dy = -k(0)f_1(x) - \int_x^\infty f_1(y)\,dk(y-x).$$

Write $$\mu(a,b) = \max_{a\leqslant x\leqslant b} |f_1(x)|,$$

and suppose x is a point such that $|f_1(x)| > |f_1(u)|$ for all $u > x$. Since $f_1(x) \to 0$, and since it is continuous, either such points exist or $f(x)$ is everywhere zero. Then

$$\begin{aligned} k(0)|f_1(x)| &\leqslant -\mu(x,x+\xi)\int_x^{x+\xi} dk(y-x) - \mu(x+\xi,\infty)\int_{x+\xi}^\infty dk(y-x) \\ &= \mu(x,x+\xi)[k(0)-k(\xi)] + \mu(x+\xi,\infty)[k(\xi)-k(\infty)] \\ &\leqslant \mu(x,x+\xi)k(0) - k(\xi)[\mu(x,x+\xi)-\mu(x+\xi,\infty)] \\ &= |f_1(x)|k(0) - k(\xi)[|f_1(x)| - \mu(x+\xi,\infty)]. \end{aligned}$$

Hence $$|f_1(x)| \leqslant \mu(x+\xi,\infty),$$

which is contrary to the definition of x. This shows that no point such as x exists, and therefore $f(x) \equiv 0$.

Alternative proof.

The integral $\int_0^\infty k(t)f(t+x)\,dt$ converges uniformly in x, for

$$\left|\int_T^\infty k(t)f(t+x)\,dt\right| = k(T)\left|\int_T^\xi f(t+x)\,dt\right|,$$

which tends to zero uniformly in x as $T \to \infty$, since $\int^\infty f(t)\,dt$ exists.

Hence, if x_1, x_2 are any positive numbers,

$$\int_{x_1}^{x_2} dx \int_0^\infty k(t)f(t+x)\,dt = \int_0^\infty k(t)\,dt \int_{x_1}^{x_2} f(t+x)\,dx.$$

The left-hand side is zero, and so, since

$$\int_{x_1}^\infty f(t+x)\,dt = -f_1(x_1+t),$$

we get
$$\int_0^\infty k(t)f_1(x_1+t)\,dt = \int_0^\infty k(t)f_1(x_2+t)\,dt,$$
for all positive x_1, x_2.

The proof of Theorem 4 applies unaltered if the right-hand side of equation (A) is any constant, and so
$$\int_0^\infty k(t)f(x+t)\,dt = \text{constant}$$
cannot have any solution save $f(x) \equiv 0$. Since $f_1(x)$ satisfies this equation and $f_1(x) \to 0$,
$$f_1(x) \equiv 0 \text{ for all } x.$$

COROLLARY. *If* $\lim_{t\to\infty} k(t) = l > 0$, (A) *has no non-zero solution.*

For, if $f(t)$ is a solution, $\int_0^\infty k(t)f(t)\,dt$ exists, and so
$$\int_X^Y f(t)\,dt = \int_X^Y \frac{(f(t)k(t))}{k(t)}\,dt$$
$$= \frac{1}{k(Y)}\int_\xi^Y k(t)f(t)\,dt \quad (X < \xi < Y)$$
$$\to 0 \quad \text{as} \quad X, Y \to \infty,$$
and so $\int^\infty f(y)\,dy$ also exists.

5. These theorems are false without the assumption that $k(t)$ is non-increasing. For example, suppose $\{a_n\}$ is a sequence of numbers such that
$$a_{2n}/a_{2n+1} = e^{-\gamma} \quad (\gamma > 0), \qquad a_{n+2} = \theta a_n \quad (\theta < 1),$$
for all values of $n \geqslant 0$.

Then, if
$$k(t) = a_n \quad (n < t < n+1),$$
it is easily seen that for any integral m
$$f(y) = e^{-[\gamma+2m\pi i]y}$$
is a solution of (A) which tends to zero as $y \to \infty$.

If $k(t)$ is non-increasing, solutions which are bounded can exist. For example, let $\{a_n\}$ be any non-increasing sequence of positive numbers tending to zero, and let
$$k(t) = a_n \qquad (n < t < n+1).$$
Then any function of period 1, whose integral over a period is zero, is a solution of (A).

It may be added that, if $k(t)$ is a non-increasing function and (A) has periodic solutions, $k(t)$ is a step-function of the type just exemplified. To prove this, we shall note the following lemma:

If $k(t)$ is a positive non-increasing function and

$$\int_0^\infty k(t)\sin 2\pi t\, dt = 0,$$

then $k(t)$ is constant in each of the intervals $n < t < (n+1)$.

$$\int_0^\infty k(t)\sin 2\pi t\, dt = \sum_{n=0}^\infty \int_n^{n+1} k(t)\sin 2\pi t\, dt$$

$$= \sum_0^\infty \int_0^1 k(t+n)\sin 2\pi t\, dt$$

$$= \sum_0^\infty \int_0^{\frac{1}{2}} \{k(t+n)-k(t+n+\tfrac{1}{2})\}\sin 2\pi t\, dt.$$

Now $k(t+n) \geqslant k(t+n+\frac{1}{2})$ for all t and all n, and $\sin t \geqslant 0$ for all t in $(0,\pi)$. Hence, if $k(t+n) \neq k(t+n+\frac{1}{2})$ for some n and some t in $(0,\frac{1}{2})$, the integrand is positive over a set of positive measure and so the integral must be positive. It is easily seen that, if a is any real number, $k(t)$ is non-increasing and positive, and $\int_0^\infty k(t)e^{iat}\,dt$ is zero, then $k(t)$ is constant in each of the intervals $(2n\pi/a,\ 2(n+1)\pi/a)$.

THEOREM 6. *If $k(t)$ is non-increasing and positive, and the equation (A) has a solution which is of period a and integrable over a period, $k(t)$ is constant in each of the intervals $2na < t < 2(n+1)a$.*

We may suppose $a = 2\pi$. We make use of the following theorem.*

If $f(x)$ is integrable and of period 2π, $k(t)$ is of bounded variation over $(0,\infty)$ and tends to zero as $t \to \infty$, and $\int_0^{2\pi} f(x)\,dx = 0$, then

$$\int_0^\infty k(t)f(t)\,dt = \sum_{n=-\infty}^\infty c_n \int_0^\infty k(t)e^{int}\,dt,$$

where
$$c_n = \frac{1}{2\pi}\int_0^{2\pi} f(x)e^{-inx}\,dx.$$

* Hardy, *Messenger of Math.* 51 (1922), 186–92. Cf. Zygmund, *Trigonometrical Series*, 92.

We have already shown that $k(t)$ must tend to zero for a solution of (A) to exist. To apply the theorem it remains to show that $c_0 = 0$. According to the alternative proof of Theorem 5

$$\int_0^\infty k(t)\,dt \int_0^{2\pi} f(t+x)\,dx = 0,$$

and so

$$c_0 \int_0^\infty k(t)\,dt = 0.$$

Since the integral is not zero, $c_0 = 0$. We can now apply the above theorem with $f(t+x)$ in place of $f(t)$. Here

$$c_n' = \frac{1}{2\pi}\int_0^{2\pi} f(t+x)e^{-int}\,dt = c_n e^{inx},$$

and so

$$\begin{aligned}\int_0^\infty k(t)f(t+x)\,dt &= \sum_{n=-\infty}^{\infty} c_n e^{inx} \int_0^\infty k(t)e^{int}\,dt \\ &= \frac{1}{\sqrt{(2\pi)}} \sum_{-\infty}^{\infty} K(n)c_n e^{inx} \\ &= 0 \quad \text{for all } x.\end{aligned}$$

Hence $c_n K(n) = 0$ for all n, since the series is convergent for all x. Therefore $K(n) = 0$, for any value of n for which $c_n \neq 0$, and so $k(t)$ is constant in each of the intervals of length $|2\pi/n|$ starting at the origin, $k(t)$ will be constant in steps of length $2\pi\eta$, where $1/\eta$ is the (positive) highest common factor of the values of n for which $c_n \neq 0$. But $2\pi\eta$ must be the period of $f(t)$; and so $k(t)$ is constant in steps of length 2π.

In conclusion, I wish to express my sincere thanks to Professor Titchmarsh, for suggesting the subject of this paper and for a great deal of advice and help in the preparation of it, and also for permission to include Theorem 5.

SOME ELEMENTARY TAUBERIAN THEOREMS (I)

By R. RADO (*Sheffield*)

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J. MERCER* proved the following theorem which has applications in the theory of Hölder and Cesàro means.

If $\alpha > 0$ *and*

$$\lim_{n\to\infty}\left[\alpha x_n+(1-\alpha)\frac{1}{n}\sum_{\nu=1}^{n} x_\nu\right] = \lambda,$$

then

$$\lim_{n\to\infty} x_n = \lambda.$$

This result may be considered as a Tauberian theorem for the linear transformation

$$y_n = \alpha x_n + \sum_{\nu=1}^{n}\frac{1-\alpha}{n}x_\nu \quad (n = 1, 2,...).$$

In particular, this Tauberian theorem does not require any 'Tauberian condition'. I. Schur† calls transformations of this type, i.e. transformations for which $y_n \to \lambda$ implies $x_n \to \lambda$, *reversible.*

In this note I shall prove some elementary Tauberian theorems, with and without Tauberian conditions, for more general classes of transformations which will include Mercer's result. I shall prove, for instance, the following theorem.

Suppose that the transformation

$$y_n = \sum_{\nu=1}^{n} a_{n\nu} x_\nu \quad (n = 1, 2,...) \tag{1}$$

is consistent, i.e. has the property that $x_n \to \xi$ *implies* $y_n \to \xi$.‡ *Also, suppose that, for every sufficiently large n and some constant* $\vartheta < 1$,

$$\sum_{\nu=1}^{n-1} |a_{n\nu}| \leqslant \vartheta |a_{nn}|. \tag{2}$$

Then (1) *is reversible.*

* *Proc. London Math. Soc.* 5 (1907), 206. G. H. Hardy (*Quart. J. of Math.* 43, 143) extended the result to complex α whose real part $\Re(\alpha)$ is positive, and I. Schur (*Math. Annalen*, 74, 447) proved that it does not hold for any α with $\Re(\alpha) \leqslant 0$. The theorems considered by Hardy and Mercer differ unessentially (αx_{n+1} instead of αx_n) from the theorem quoted.

† Loc. cit.

‡ If a transformation T has an inverse T^{-1}, then the statements: (i) T is consistent, (ii) T^{-1} is reversible, are equivalent.

This shows, for instance, that in the space of consistent linear transformations a certain neighbourhood of the identical transformation contains only reversible transformations. In the special case considered by Mercer, (2) becomes

$$|1-\alpha|\frac{n-1}{n} \leqslant \vartheta\left|\alpha+\frac{1-\alpha}{n}\right|.$$

This inequality is satisfied for a given α, for every sufficiently large n and some constant $\vartheta < 1$, if and only if $|(1-\alpha)/\alpha| < 1$, i.e. $\mathfrak{R}(\alpha) > \frac{1}{2}$. The remaining case of Mercer's theorem $(0 < \alpha \leqslant \frac{1}{2})$ follows from another theorem (Theorem 6) of this note.

The proofs are very elementary. Every theorem for sequences has its integral analogue for functions which will be considered in a second paper.

We are concerned with a transformation

$$y_n = \sum_{\nu=1}^{\infty} a_{n\nu} x_\nu \quad (n = 1, 2, \ldots). \tag{3}$$

If not stated otherwise, every number which occurs is complex. Relations such as

$$x_n \to 0; \qquad x_n = O(F_n); \qquad y_n \to 0$$

refer to the limit process $n \to \infty$.

For convenience we enumerate some hypotheses which we shall use.

$$\varliminf_{n\to\infty} |a_{nn}| > 0, \tag{4}$$

$$\varlimsup_{n\to\infty} \sum_{\nu\neq n} \left|\frac{a_{n\nu}}{a_{nn}}\right| \leqslant \vartheta < 1, \tag{5}$$

$$\lim_{n\to\infty} a_{n\nu}/a_{nn} = 0 \quad (\nu = 1, 2, \ldots), \tag{6}$$

$$a_{n\nu} = 0 \quad \text{for} \quad n < \nu, \tag{7}$$

$$\sum_{\nu\neq n} \left|\frac{a_{n\nu}}{a_{nn}}\right| \leqslant \vartheta < 1 \tag{8}$$

for every sufficiently large n,*

$$a_{n\nu} = 0 \quad \text{for} \quad n+k < \nu. \tag{9}$$

* In particular, any of the relations (5), (6), (8) implies that, for every sufficiently large n, $a_{nn} \neq 0$ and, in the cases (5), (8), the series $\sum_{\nu=1}^{\infty} |a_{n\nu}|$ converges.

THEOREM 1. *If* (4), (5), (6) *hold, then*

$$y_n \to 0; \qquad x_n = O(1)$$

imply

$$x_n \to 0.$$

This statement is no longer true if the condition $x_n = O(1)$ *is replaced by*

$$x_n = O(F_n), \tag{10}$$

where F_n *is an arbitrary sequence of positive numbers which may depend on* ϑ *and satisfy*

$$\overline{\lim} F_n = \infty. \tag{11}$$

THEOREM 2. *If* (4), (5), (7) *hold, then* $y_n = O(1)$ *implies* $x_n = O(1)$.

THEOREM 3. *If* (4), (5), (6), (7) *hold, then* $y_n \to 0$ *implies* $x_n \to 0$.

THEOREM 4. *If* (4), (8), (9) *hold for some positive integer* k,

then

$$y_n = O(1)$$

and

$$x_n = o(\vartheta^{-n/k}) \tag{12}$$

imply

$$x_n = O(1).$$

This statement is no longer true if either (8) *is replaced by* (5), *or* (12) *by*

$$x_n = O(\vartheta^{-n/k}). \tag{13}$$

THEOREM 5. *If* (4), (6), (8), (9) *hold for some positive integer* k,

then

$$y_n \to 0,$$

and

$$x_n = o(\vartheta^{-n/k}) \tag{14}$$

imply

$$x_n \to 0.$$

This statement is no longer true if either (8) *is replaced by* (5), *or* (14) *by* (13).

THEOREM 6. *Consider a transformation of the form*

$$y_n = a_n x_n + b_n \sum_{\nu=1}^{n} c_\nu x_\nu \quad (n = 1, 2, ...), \tag{15}$$

where the a_n *are real, and where*

$$\underline{\lim}\, a_n > 0, \qquad b_n > 0, \qquad b_n \to 0,$$

$$b_n \leqslant C b_{n'} \quad \text{if} \quad n' \leqslant n, \tag{16}$$

C being some constant, $\qquad c_\nu \geqslant 0.$

Then $y_n \to 0$ *implies* $x_n \to 0$.

If, in addition to the conditions stated,

$$\lim_{n\to\infty} \sum_{\nu=1}^{\infty} a_{n\nu} = 1,$$

then in the wording of Theorems 1, 3, 5, 6 the relations

$$x_n \to 0, \qquad y_n \to 0$$

may be replaced respectively by

$$x_n \to \lambda, \qquad y_n \to \lambda,$$

where λ is some arbitrary number. For then $y_n \to \lambda$ implies

$$\sum_{\nu=1}^{\infty} a_{n\nu}(x_\nu - \lambda) \to 0,$$

$$(x_n - \lambda) \to 0, \qquad x_n \to \lambda.$$

The case $\alpha > \frac{1}{2}$ of Mercer's theorem follows from Theorem 3, as indicated in the introduction. The more general theorem mentioned in the introduction is also a corollary of Theorem 3. To obtain Mercer's theorem for $0 < \alpha \leqslant \frac{1}{2}$, put, in Theorem 6,

$$a_n = \alpha, \qquad b_n = (1-\alpha)/n, \qquad c_\nu = 1.$$

Then (16) holds, with $C = 1$.

In proving Theorems 1 to 5 we may assume, without loss of generality, that

$$a_{nn} \neq 0 \quad \text{for every } n.$$

For, by (4), there are only a finite number of indices n for which $a_{nn} = 0$, and, if we alter the corresponding numbers a_{nn}, we do not affect the validity of our hypotheses. If we now write (3) in the form

$$y'_n = \sum_{\nu} a'_{n\nu} x_\nu,$$

where

$$y'_n = y_n/a_{nn} \qquad a'_{n\nu} = a_{n\nu}/a_{nn},$$

we see that the numbers y'_n, $a'_{n\nu}$ satisfy the same conditions as the numbers y_n, $a_{n\nu}$. We have $a'_{nn} = 1$. Hence we may assume, without loss of generality, that

$$a_{nn} = 1 \quad (n = 1, 2, \ldots).$$

Proof of Theorem 1. Let us suppose that

$$0 < \overline{\lim}\,|x_n| = r < \infty. \tag{17}$$

Given $\epsilon > 0$, we can find an integer $n_0(\epsilon) > 0$ such that, for every $n > n_0$,

$$|y_n| \leqslant \epsilon, \qquad |x_n| \leqslant r+\epsilon, \qquad \sum_{\nu \neq n} |a_{n\nu}| \leqslant \vartheta+\epsilon.$$

There are arbitrarily large numbers $n_1 > n_0$ for which $|x_{n_1}| \geqslant r-\epsilon$. Then

$$r-\epsilon \leqslant |x_{n_1}| = \Big|y_{n_1} - \sum_{\nu \neq n_1} a_{n_1\nu} x_\nu\Big| \leqslant |y_{n_1}| + \Big|\sum_{\nu=1}^{n_0} a_{n_1\nu} x_\nu\Big| + \Big|\sum_{\substack{\nu > n_0 \\ \nu \neq n_1}} a_{n_1\nu} x_\nu\Big|$$

$$\leqslant \epsilon + \Big|\sum_{\nu=1}^{n_0} a_{n_1\nu} x_\nu\Big| + (r+\epsilon)(\vartheta+\epsilon).$$

Make $n_1 \to \infty$. We obtain, by (6),

$$r-\epsilon \leqslant \epsilon+0+(r+\epsilon)(\vartheta+\epsilon).$$

Let $\epsilon \to 0$. Then $$r \leqslant r\vartheta,$$

i.e. $$1 \leqslant \vartheta.$$

This contradicts (5). Hence (17) is not true, and the positive part of the theorem is proved.

To show that the theorem is a best possible one in the sense stated, we choose integers $k_1, k_2,...$ $(0 < k_1 < k_2 < ...)$ and put

$$y_{k_m} = x_{k_m} - \vartheta x_{k_{m+1}} \quad (m = 1, 2, ...),$$
$$y_n = x_n \qquad (n \neq k_m),$$

where ϑ is an arbitrary number such that $0 < \vartheta < 1$. Put

$$x_{k_m} = \vartheta^{-m} \quad (m = 1, 2, ...),$$
$$x_n = 0 \qquad (n \neq k_m).$$

Then $y_n = 0$, but $x_n \to 0$ is not true. (10) is the same as

$$\vartheta^{-m} = O(F_{k_m}) \quad (m \to \infty).$$

Given a sequence $F_1, F_2,...$ satisfying (11), we determine the k_m as follows. Put $k_1 = 1$. Suppose that, for some $m > 1$, $k_1, k_2,..., k_{m-1}$ have been defined already. Then, by (11), there is a least index k' such that

$$k' > k_{m-1}, \qquad F_{k'} \geqslant \vartheta^{-m}.$$

Put $k_m = k'$. Then the example shows that (10) is not a 'Tauberian condition' for our transformation.

Proof of Theorem 2. We have, for a suitable constant C,

$$|y_n| \leqslant C \quad \text{for every } n > 0.$$

Let us suppose that, contrary to the assertion of the theorem,

$$\overline{\lim}\, |x_n| = \infty.$$

Choose $\epsilon > 0$. For some arbitrary $A > 0$, let $n_1 = n_1(A)$ be the least index ν for which $|x_\nu| \geqslant A$. Then

$$\lim_{A\to\infty} n_1 = \infty.$$

Hence, for every $A \geqslant A_0(\epsilon)$,

$$\sum_{\nu=1}^{n_1-1} |a_{n_1\nu}| \leqslant \vartheta+\epsilon.$$

For these A, we have

$$|x_{n_1}| \leqslant |y_{n_1}| + \left|\sum_{\nu=1}^{n_1-1} a_{n_1\nu} x_\nu\right| \leqslant C + |x_{n_1}|(\vartheta+\epsilon),$$

$$1 \leqslant \frac{C}{|x_{n_1}|} + \vartheta + \epsilon \leqslant \frac{C}{A} + \vartheta + \epsilon.$$

Let $A \to \infty$. Then $1 \leqslant 0+\vartheta+\epsilon$. Make $\epsilon \to 0$. Then $1 \leqslant \vartheta$.

This contradiction proves the theorem.

Proof of Theorem 3. Since $y_n \to 0$, we have, *a fortiori*, $y_n = O(1)$. Therefore, by Theorem 2, $x_n = O(1)$, and, by Theorem 1, $x_n \to 0$.

Proof of Theorem 4. It is no loss of generality to assume that, for *every* n,

$$\sum_{\nu \neq n} |a_{n\nu}| \leqslant \vartheta.$$

There is a C such that, for every n,

$$|y_n| \leqslant C.$$

Put
$$\epsilon_n = \vartheta^{n/k}|x_n|.$$

Then $\epsilon_n \to 0$. We shall prove that, for every n,

$$|x_n| \leqslant C(1-\vartheta)^{-1}. \tag{18}$$

This inequality would prove the positive part of the theorem.

Suppose that $n = n_0$ is the least index for which (18) is not true. Then
$$|x_\nu| \leqslant C(1-\vartheta)^{-1} < |x_{n_0}| \quad \text{for} \quad 1 \leqslant \nu \leqslant n_0.$$

Corresponding to every $n > 0$, we define n' as being the least integer such that
$$1 \leqslant n' \leqslant n+k,$$
$$|x_\nu| \leqslant |x_{n'}| \quad \text{for} \quad 1 \leqslant \nu \leqslant n+k.$$

Then
$$|x_n| \leqslant |y_n| + \left|\sum_{\nu\neq n} a_{n\nu} x_\nu\right|$$
$$\leqslant C + |x_{n'}|\vartheta. \tag{19}$$

Put
$$n_1 = (n_0)', \qquad n_2 = (n_1)',$$
and generally,
$$n_{m+1} = (n_m)' \quad (m = 0, 1, 2, \ldots).$$
Then, obviously,
$$n_0 \leqslant n_1 \leqslant n_2 \leqslant \ldots,$$
$$n_m \leqslant n_0 + km.$$

Using (19) repeatedly we obtain

$$|x_{n_0}| \leqslant C + \vartheta|x_{n_1}| \leqslant C + \vartheta C + \vartheta^2 |x_{n_2}|$$
$$\leqslant \ldots \leqslant C(1+\vartheta+\vartheta^2+\ldots) + \vartheta^m |x_{n_m}|.$$

We shall have obtained a contradiction with the definition of n_0 if we prove that

$$\lim_{m\to\infty} \vartheta^m |x_{n_m}| = 0.$$

Now, if $n_m \to \infty$ as $m \to \infty$, we have

$$\vartheta^m |x_{n_m}| = \vartheta^m \epsilon_{n_m} \vartheta^{-n_m/k} \leqslant \vartheta^m \epsilon_{n_m} \vartheta^{-(n_0+km)/k} \to 0$$

as $m \to \infty$. On the other hand, if, for some $l \geqslant 0$,

$$n_l = n_{l+1} = \dots,$$

we have, for every $m \geqslant l$,

$$\vartheta^m |x_{n_m}| = \vartheta^m |x_{n_l}| \to 0$$

as $m \to \infty$. The positive part of Theorem 4 is proved.

In order to show that (8) cannot be replaced by (5), we choose a number ϑ $(0 < \vartheta < 1)$ and an integer $k > 0$ and consider the transformation

$$y_n = x_n - \vartheta(n+k)n^{-1}x_{n+k}.$$

We put

$$x_n = n^{-1}\vartheta^{-n/k}.$$

Then

$$y_n = n^{-1}\vartheta^{-n/k} - \vartheta(n+k)n^{-1}(n+k)^{-1}\vartheta^{-(n+k)/k} = 0.$$

In this case (4), (5), (9) are satisfied and

$$y_n = O(1), \qquad x_n = o(\vartheta^{-n/k}),$$

but $x_n = O(1)$ is not true.

The example

$$y_n = x_n - \vartheta x_{n+k}, \qquad x_n = \vartheta^{-n/k}, \qquad y_n = 0$$

shows that (12) cannot be replaced by (13).

Proof of Theorem 5. Since $y_n = O(1)$, Theorem 4 shows that $x_n = O(1)$. Now Theorem 1 is applicable, for (8) implies (5); and we obtain $x_n \to 0$.

The same examples as were used in the proof of Theorem 4 show that Theorem 5 is a best possible one in the sense stated.

Before proving Theorem 6 we should notice that some condition on b_n besides $b_n > 0$, $b_n \to 0$ is necessary in order that the conclusion of Theorem 6 may hold, as is shown by the following example.

$$a_n = 1 \quad \text{for all } n,$$

$$b_n = x_n = 1/n; \quad c_n = 1 \quad \text{if } n \text{ is not a power of 2,}$$

$$b_n = (\log n + 1)^{-1}; \quad c_n = 0; \quad x_n = -b_n \sum_{1}^{n-1} c_\nu x_\nu \quad \text{if } n \text{ is a power of 2.}$$

Here $y_n \to 0$ holds, but not $x_n \to 0$, since

$$x_{2^m} = -[\log(2^m)+1]^{-1}\Big(\sum_{\nu=1}^{2^m} \nu^{-1} - \sum_{\mu=0}^{m} 2^{-\mu}\Big) \to -1$$

as $m \to \infty$.

In the case of real sequences x_n the theorem remains true if (15) is replaced by

$$y_n = a_n x_n + b_n \sum_1^n c_\nu x_\nu^{u_\nu} \tag{20}$$

where the u_ν are arbitrary odd integers, $u_\nu > 0$ whenever $x_\nu = 0$. For (20) can be written as

$$y_n = a_n x_n + b_n \sum_1^n c'_\nu x_\nu$$

where $$c'_\nu = c_\nu x_\nu^{u_\nu - 1} \geqslant 0.$$

In proving Theorem 6 we may restrict ourselves to the case of real sequences x_n. The general case is obtained by applying this special result to the real and to the imaginary parts of the x_n separately. Since

$$y_n - a_n x_n - b_n \sum_2^n c_\nu x_\nu = b_n c_1 x_1 \to 0$$

as $n \to \infty$, it is no loss of generality to assume that $x_1 = 0$.

Corresponding to every integer $n > 0$ we define $n' = n'(n)$ as being the largest integer ν satisfying

$$1 \leqslant \nu \leqslant n, \qquad x_\nu \leqslant 0.$$

It is sufficient to prove the following lemma.

LEMMA. *Suppose the hypotheses of Theorem* 6 *are satisfied. Furthermore, suppose that n tends to infinity through a sequence S which has the property that*

$$a_n > 0, \quad x_n \geqslant 0 \quad \textit{for every } n \textit{ of } S,$$

and either $$n' = n'_0 = \text{constant} \quad \textit{for every } n \textit{ of } S$$

or $$n' \to \infty \quad \textit{as } n \to \infty \; (n \textit{ in } S).$$

Then $$\lim_{\substack{n \to \infty \\ n \text{ in } S}} x_n = 0. \tag{21}$$

For, if Theorem 6 were not true, then we should have, for some real sequence x_n satisfying the hypotheses of the theorem, either

$$\overline{\lim}\, x_n > 0, \tag{22}$$

or $$\underline{\lim}\, x_n < 0$$

(or both). After multiplying (15) by -1 if necessary, and con-

sidering $-x_n$, $-y_n$ as variables, we obtain a case where (22) holds. Then we can find a sequence S such that

$$\lim_{\substack{n\to\infty \\ n \text{ in } S}} x_n = \xi > 0 \quad (\xi \text{ finite or } +\infty).$$

If we replace S by a suitable sub-sequence we can obviously satisfy the conditions laid down for S. Then (21) would not be true.

For the rest of this proof n is always a number of S, and limiting processes refer to $n \to \infty$ (n in S). Using the definition of n', we have

$$\begin{aligned} 0 \leqslant a_n x_n = y_n - b_n \sum_1^n c_\nu x_\nu &\leqslant y_n - b_n \sum_1^{n'} c_\nu x_\nu \\ = y_n - \frac{b_n}{b_{n'}}(y_{n'} - a_{n'} x_{n'}) &\leqslant y_n - \frac{b_n}{b_{n'}} y_{n'}. \end{aligned}$$

If $n' = n'_0 =$ constant, then

$$\frac{b_n}{b_{n'}} y_{n'} = b_n \times \text{constant} \to 0.$$

If $n' \to \infty$, then $$\left|\frac{b_n}{b_{n'}} y_{n'}\right| \leqslant C|y_{n'}| \to 0.$$

Hence, in either case,

$$y_n - \frac{b_n}{b_{n'}} y_{n'} \to 0, \qquad a_n x_n \to 0, \qquad x_n = \frac{a_n x_n}{a_n} \to 0,$$

and the theorem is proved.

THE LORENTZ TRANSFORMATION AND THE DUAL NATURE OF LIGHT

By G. TEMPLE (*London*)

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1. Introduction

THE object of this note is to give yet another discussion of the Lorentz transformation, which appears, in some respects, to be more fundamental than the customary treatment. The Lorentz transformation gives the relation between the spatial and temporal measurements made by two observers in uniform relative motion. In the usual account of the transformation there is an obvious but remarkable change in the status of the two related observers from the theoretical deduction of the transformation to its practical applications. In the theoretical deduction the two observers are treated as equivalent, although this equivalence is expressed by different writers in different ways. Thus, for example, Einstein assumed an optical equivalence such that, in the space of each observer, the speed of light was the same in all directions. But, in the practical application of the transformation, one observer is identified with an actual physicist in a terrestrial laboratory while the other observer is a fictitious character who may be supposed to be carried about with a high-speed electron or with a distant nebula. This description of the changed status of the observers is not intended as a criticism of the customary discussion. The fictitious character of the second observer does not invalidate the Lorentz transformation, whose practical utility arises precisely from the fact that it enables us to infer what measurements the second observer would make from the actual measurements made by the first observer.

Nevertheless, in view of the different status of the observers in the practical application of the transformation, it is arguable that there is room for a theoretical discussion in which as little emphasis as possible is laid upon the equivalence of observers. All that we shall here assume initially regarding the equivalence of observers is that the optical situation which appears as a parallel beam of light to one observer will appear as a parallel beam of light to the other observer, and even this assumption will be abandoned later (§ 6). We

shall also assume that for one observer Σ the speed of light is the same in all directions. As regards the other observer S we shall make no assumption regarding the isotropy of the speed of light in his system of reference. This asymmetry in our assumptions corresponds to the distinction between our considerable knowledge of the optical experience of observers stationary in fixed laboratories and our very meagre knowledge of the optical experience of observers in relative motion.

Our analysis of the relations between the two observers depends upon the recognition of the dual character of a beam of light. A parallel beam of light can be regarded as a system of plane waves advancing normal to themselves with velocity $\mathbf{w}$, or as a system of photons advancing along parallel rectilinear rays with velocity $\mathbf{v}$. Only the most primitive and elementary concepts of the wave theory and the particle theory of light are involved in these two descriptions. These two representations of a parallel beam of light will be consistent only if the wave velocity $\mathbf{w}$ and the ray velocity $\mathbf{v}$ agree in magnitude and direction.† It is this principle of consistency which is the basis of our deduction of the Lorentz transformation.

From this principle we shall deduce that, if each of our observers employs Galilean coordinates, then the equations of transformation from one system of coordinates to another are *linear*, so that the observers have no relative acceleration. Next we prove that, if the speed of light is the same in all directions for the observer Σ, then the same is also true for the observer S. Finally, we show that the transformation must be of the Lorentzian type, and that the speeds of light as measured by S and Σ are in the same ratio as the speed of Σ as measured by S to the speed of S as measured by Σ.

2. The Dual Representation of a Beam of Light

Let the Galilean systems of coordinates employed by S and Σ be (x_1, x_2, x_3, x_4) and $(\xi_1, \xi_2, \xi_3, \xi_4)$, in which (x_j) and (ξ_j) $(j = 1, 2, 3)$ are each rectangular spatial coordinates, while x_4 and ξ_4 are temporal coordinates. We shall habitually allow italic suffixes i, j, k, p, q to take the values 1, 2, 3, and Greek suffixes, μ, ν, ρ, to take the values 1, 2, 3, 4. The summation convention will not be employed, as it would need to be frequently suspended.

Let a parallel beam of light P have direction cosines $\lambda_1, \lambda_2, \lambda_3$ in

† We are, of course, considering a beam of light *in vacuo*.

the system Σ and the speed γ, which is assumed to be the same in all directions. Then P can be regarded either as the system of plane waves,

$$\sum \lambda_k \xi_k - \gamma \xi_4 = \alpha \text{ (a constant)}, \tag{2.1}$$

or as the system of rectilinear rays,

$$\xi_k - \lambda_k \gamma \xi_4 = \alpha_k \text{ (a constant)}. \tag{2.2}$$

Similarly, in the system S, P can be regarded either as the system of plane waves,

$$\sum l_k x_k - c(l) x_4 = a, \tag{2.3}$$

or as the system of rectilinear rays,

$$x_k - l_k c(l) x_4 = a_k, \tag{2.4}$$

but here the speed $c(l)$ or $c(l_1, l_2, l_3)$ is to be treated as an unknown function of the direction cosines l_1, l_2, l_3.

Let the equations of transformation connecting the coordinates of S and Σ be

$$\xi_\mu = \phi_\mu(x_1, x_2, x_3, x_4) \equiv \phi_\mu(x).$$

The functions ϕ_μ cannot depend in any way upon the direction cosines l_k or λ_k which specify a beam of light P. We know, however, that the equations (2.1) and (2.3) represent the same system of wave fronts. Hence the equations

$$\phi(x) \equiv \sum \lambda_j \phi_j(x) - \gamma \phi_4(x) = \alpha$$

and

$$u \equiv \sum l_j x_j - c(l) x_4 = a$$

represent the same system of surfaces. Therefore

$$\alpha = F(a, \lambda) \quad \text{and} \quad \phi(x) = F(u, \lambda). \tag{2.5}$$

We can now proceed to prove that the functions $\phi_\mu(x)$ are linear functions of x_ν.

3. The Transformation is Linear

The direction cosines of P in S, viz. l_1, l_2, l_3, are functions of the direction cosines of P in Σ, viz. λ_1, λ_2, λ_3. When these latter have the special values

$$\lambda_k = \delta_{pk} = \begin{cases} 0 \ (k \neq p), \\ 1 \ (k = p), \end{cases}$$

we shall denote by*

$$l_{pk},\ c_p, \quad \text{and} \quad u_p \equiv \sum l_{pk} x_k - c_p x_4 \tag{3.1}$$

the corresponding values of

$$l_k,\ c(l), \quad \text{and} \quad u.$$

We shall also write $F_p(u_p)$ for the corresponding form of the function $F(u,\lambda)$. It then follows from (2.5) that

$$\phi_p(x)-\gamma\phi_4(x) = F_p(u_p), \tag{3.2}$$

and on substituting the values of $\phi_p(x)$ given by this equation in (2.5), it also follows that

$$\gamma(\sum \lambda_p-1)\phi_4(x) = F(u,\lambda)-\sum \lambda_p F_p(u_p). \tag{3.3}$$

So far we have used only the equivalence of (2.1) and (2.3). We now employ the fact that (2.2) and (2.4) represent the same system of curves. It follows that the two sets of equations

$$\phi_j(x)-\lambda_j\gamma\phi_4(x) = \alpha_j \quad \text{and} \quad x_j-l_j c(l)x_4 = a_j$$

are equivalent. Hence

$$\sum \frac{\partial\phi_j}{\partial x_k} l_k c(l)+\frac{\partial\phi_j}{\partial x_4} = \lambda_j\gamma\left\{\sum \frac{\partial\phi_4}{\partial x_k} l_k c(l)+\frac{\partial\phi_4}{\partial x_4}\right\}.$$

Into this equation we substitute the value of $\phi_p(x)$ given by (3.2), and thus obtain

$$\gamma(1-\lambda_p)\left\{\sum \frac{\partial\phi_4}{\partial x_k} l_k c(l)+\frac{\partial\phi_4}{\partial x_4}\right\} = -G_p,$$

where $$G_p = \sum \frac{\partial F_p(u_p)}{\partial x_k} l_k c(l)+\frac{\partial F_p(u_p)}{\partial x_4}. \tag{3.4}$$

We now substitute the value of $\phi_4(x)$ given by (3.3). Then, using (2.5),

$$\gamma(1-\lambda_p)\sum \lambda_q G_q = \gamma(\sum \lambda_q-1)G_p.$$

It thus appears that

$$\frac{G_1}{1-\lambda_1} = \frac{G_2}{1-\lambda_2} = \frac{G_3}{1-\lambda_3}, \tag{3.5}$$

where, by (3.4) and (3.1),

$$G_p = \{c(l)\sum l_{pk} l_k-c_p\}F_p',$$

F_p' being the ordinary derivative $dF_p(u_p)/du_p$. We shall now prove that these derivatives are mere constants.

We choose l_1, l_2, l_3 so that

$$\sum l_k^2 = 1 \quad \text{and} \quad \sum l_{pk} l_k = 0 \quad \text{for} \quad p = 2 \quad \text{and} \quad p = 3.$$

Then $$G_2 = -c_2 F_2' \quad \text{and} \quad G_3 = -c_3 F_3'.$$

The direction cosines λ_1, λ_2, λ_3 are functions of l_1, l_2, l_3 and will assume special values under these conditions. The actual values assumed are not required, but we must show that $\lambda_2 \neq 1$ and $\lambda_3 \neq 1$.

For, if we assume that $\lambda_2 = 1$, then $\lambda_1 = 0$ and $\lambda_3 = 0$. Hence, by (3.1),

$$l_k = l_{2k}, \qquad c(l) = c_2, \quad \text{and} \quad G_2 = c_2\{\sum l_{2k}^2 - 1\}F_2'.$$

But, since $\sum l_{2k} l_k$ becomes $\sum l_{2k}^2$, it is clear that the value of the latter expression is zero. Hence $G_2 = -c_2 F_2'$. But equations (3.5) show that

$$\frac{G_2}{0} = \frac{G_3}{1},$$

which is impossible. Hence λ_2 cannot equal unity. A similar proof holds for λ_3.

It follows that equations (3.5) yield a non-degenerate relation

$$\frac{c_2 F_2'}{1-\lambda_2} = \frac{c_3 F_3'}{1-\lambda_3},$$

where the denominator of neither fraction can vanish. Hence F_2' is a numerical multiple of F_3', and, unless both derivatives are constants, u_2 will be a function of u_3. Equation (3.2) then shows that $\xi_2 - \gamma\xi_4$ will be a function of $\xi_3 - \gamma\xi_4$, which is absurd. Therefore F_2' and F_3' are mere constants, and the same is true of F_1'.

We shall write $F_p' = a_p$, whence

$$\phi_p(x) - \gamma\phi_4(x) = a_p u_p + b_p,$$

where b_p is a constant. We have thus shown that

$$\xi_p - \gamma\xi_4 = a_p\{\sum l_{pk} x_k - c_p x_4\} + b_p, \tag{3.6}$$

and a similar argument would prove that

$$x_q - \bar{c}_q x_4 = \alpha_q\{\sum \lambda_{qj}\xi_j - \gamma\xi_4\} + \beta_q, \tag{3.7}$$

where $\bar{c}_q \lambda_{qj}$ are the values assumed by $c(l), \lambda_j$ when $l_k = \delta_{qk}$. If we eliminate ξ_1, ξ_2, ξ_3 between these equations, we obtain a relation of the form

$$\alpha_q \gamma(1 - \sum_j \lambda_{qj})\xi_4 = \text{a linear function of } x_1, x_2, x_3, x_4. \tag{3.8}$$

It is, however, necessary to prove that this relation is effective, i.e. that the coefficients of ξ_4, x_1, x_2, x_3, x_4 do not all vanish.

To prove this we note that, if α_q were to vanish, then by (3.7) we should have an identical relation connecting x_q and x_4, which is impossible. Therefore, if the coefficient of ξ_4 vanishes in (3.8), we must have

$$\sum_j \lambda_{qj} = 1. \tag{3.9}$$

But, by a suitable choice of the system S, the direction cosines $(\lambda_{q1}, \lambda_{q2}, \lambda_{q3})$ can be given any prescribed values subject to

$$\lambda_{q1}^2+\lambda_{q2}^2+\lambda_{q3}^2 = 1.$$

Hence the equation (3.9) cannot be valid.

Hence, in (3.8), the coefficient of ξ_4 does not vanish. The right-hand side of this equation cannot, therefore, be a mere constant. Hence

$$\xi_4 = \text{a linear function of } x_1,\ x_2,\ x_3,\ x_4.$$

It then follows from (3.6) that

$$\xi_p = \text{a linear function of } x_1,\ x_2,\ x_3,\ x_4.$$

We have thus proved that the transformation is necessarily linear.

4. The Speed of Light in S is the same in all directions

To show that in the system S the speed of light is the same in all directions we shall make a fresh start and write the equations of transformation in the form

$$\xi_\mu = \sum_\nu \beta_{\mu\nu} x_\nu + \text{constant}.$$

It follows that

$$\sum \lambda_j \xi_j - \gamma\xi_4 = \sum_k \Big\{\sum_j \lambda_j \beta_{jk} - \gamma\beta_{4k}\Big\} x_k + \sum_j (\lambda_j \beta_{j4} - \gamma\beta_{44}) x_4 + \text{constant}.$$

Hence, from a comparison of equations (2.1) and (2.3),

$$\frac{l_k}{c(l)} = \frac{\sum_j \lambda_j \beta_{jk} - \gamma\beta_{4k}}{-\sum_j \lambda_j \beta_{j4} + \gamma\beta_{44}}. \tag{4.1}$$

Also, if $x_j = l_j c(l) x_4$, then

$$\xi_j = \Big\{\sum_k \beta_{jk} l_k c(l) + \beta_{j4}\Big\} x_4 + \text{constant},$$

and

$$\xi_4 = \Big\{\sum_k \beta_{4k} l_k c(l) + \beta_{44}\Big\} x_4 + \text{constant}.$$

Therefore, from a comparison of equations (2.2) and (2.4),

$$\lambda_j \gamma = \frac{\sum_i \beta_{ji} l_i c(l) + \beta_{j4}}{\sum_i \beta_{4i} l_i c(l) + \beta_{44}}. \tag{4.2}$$

Equations (4.1) and (4.2) determine the law of transformation of the direction cosines (l_k) and (λ_k). If we substitute the values of λ_j

given by (4.2) in (4.1), we obtain relations of the form

$$\frac{l_k}{c(l)} = \frac{\sum_i B_{ki} l_i c(l) + A_k}{-\sum_i A_i l_i c(l) + A}, \tag{4.3}$$

where
$$\left.\begin{aligned} B_{ki} &= \sum_j \beta_{ji}\beta_{jk} - \gamma^2 \beta_{4i}\beta_{4k} \\ A_k &= \sum_j \beta_{j4}\beta_{jk} - \gamma^2 \beta_{44}\beta_{4k} \\ A &= -\sum_j \beta_{j4}\beta_{j4} + \gamma^2 \beta_{44}\beta_{44} \end{aligned}\right\}. \tag{4.4}$$
and

These relations, (4.3), are not identities, but are equations which determine $c(l)$ as a function of l_1, l_2, l_3. These three equations for $c(l)$ must be consistent, and from their consistency we shall deduce that $c(l)$ is a constant c, that $A_k = 0$, and that $B_{ki} = (A/c^2)\delta_{ki}$.

It is convenient to write the relations (4.3) in vectorial form as

$$\boldsymbol{\beta} c^2(l) + \{\boldsymbol{\alpha} + (\boldsymbol{\alpha}\,.\,\mathbf{l})\mathbf{l}\}c(l) - A\mathbf{l} = 0, \tag{4.5}$$

where $\boldsymbol{\beta}$ denotes the vector with components

$$\beta_k = \sum_i B_{ki} l_i,$$

$\boldsymbol{\alpha}$ denotes the vector with components

$$\alpha_k = A_k,$$

and $\mathbf{l}$ denotes the unit vector with components l_k.

In equation (4.5) we write $l_j = \delta_{pj}$ and $c(l) = \bar{c}_p$. Then we obtain

$$B_{kp}\bar{c}_p^2 + (\alpha_k + \alpha_p \delta_{pk})\bar{c}_p - A\delta_{pk} = 0.$$

Hence
$$\left.\begin{aligned} \beta_k = \sum_p B_{kp} l_p &= -\alpha_k \sum_p (l_p/\bar{c}_p) + B_{kk}^* l_k \\ B_{kk}^* &= A/\bar{c}_k^2 - \alpha_k/\bar{c}_k \end{aligned}\right\}. \tag{4.6}$$
where

Returning to equation (4.5), we multiply this vectorially by $\mathbf{l}$, and thus obtain

$$\boldsymbol{\beta} \wedge \mathbf{l}\, c(l) + \boldsymbol{\alpha} \wedge \mathbf{l}\, c(l) = 0.$$

Hence
$$\boldsymbol{\alpha}\,.\,\boldsymbol{\beta} \wedge \mathbf{l} = 0.$$

This last equation can be written as

$$\begin{vmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ B_{11}^* l_1 & B_{22}^* l_2 & B_{33}^* l_3 \\ l_1 & l_2 & l_3 \end{vmatrix} = 0, \tag{4.7}$$

in virtue of (4.6). Now (4.7) is one of the conditions of consistency of the three expressions for $c(l)$ given by (4.5), and it must therefore

be satisfied for all values of l_1, l_2, l_3. By putting $l_1 = 0, l_2 \neq 0, l_3 \neq 0$, we find that

$$\alpha_1(B^*_{22}-B^*_{33}) = 0;$$

and, similarly, we obtain the equations

$$\alpha_2(B^*_{33}-B^*_{11}) = 0 \quad \text{and} \quad \alpha_3(B^*_{11}-B^*_{22}) = 0.$$

There are now four possible cases to examine and we shall prove that in each case $B^*_{11} = B^*_{22} = B^*_{33}$.

(i) $\alpha_1 = 0, \alpha_2 = 0, \alpha_3 = 0$.

Then (4.5) becomes $\quad B^*_{kk}\, l_k\, c^2(l) = Al_k,$

whence $B^*_{11} = B^*_{22} = B^*_{33}$.

(ii) $\alpha_1 = 0, \alpha_2 = 0, B^*_{11} = B^*_{22} = B^*$, say.

Then (4.5) becomes

$$B^*c^2(l)+(\alpha_3 l_3)c(l)-A = 0,$$

and
$$\Big\{B^*_{33}\, l_3-\alpha_3 \sum_p (l_p/\bar{c}_p)\Big\}c^2(l)+\{\alpha_3+\alpha_3\, l_3^2\}c(l)-Al_3 = 0,$$

by (4.6). Therefore

$$\Big\{(B^*-B^*_{33})l_3+\alpha_3 \sum_p (l_p/\bar{c}_p)\Big\}c^2(l) = \alpha_3\, c(l).$$

Now put $l_3 = 1$, $l_1 = 0$, and $l_2 = 0$, and we find that

$$\{(B^*-B^*_{33})+\alpha_3/\bar{c}_3\}\bar{c}_3 = \alpha_3,$$

whence $B^*_{33} = B^*$ as before.

(iii) $\alpha_1 = 0, B^*_{33} = B^*_{11}, B^*_{11} = B^*_{22}$.

Hence we have the desired result immediately.

(iv) $B^*_{22} = B^*_{33}, B^*_{33} = B^*_{11}, B^*_{11} = B^*_{22}$.

Again the desired relation is obvious.

Hence in all cases we see from (4.6) that

$$\beta_k = -\alpha_k \sum_p (l_p/\bar{c}_p)+B^*l_k. \tag{4.8}$$

Therefore (4.5) becomes

$$\Big\{B^*\mathbf{l}-\boldsymbol{\alpha} \sum_p (l_p/\bar{c}_p)\Big\}c^2(l)+\{\boldsymbol{\alpha}+(\boldsymbol{\alpha}\,.\,\mathbf{l})\mathbf{l}\}c(l)-A\mathbf{l} = 0. \tag{4.9}$$

Multiplying vectorially by $\mathbf{l}$, we find that

$$\Big\{1-c(l) \sum_p (l_p/\bar{c}_p)\Big\}\boldsymbol{\alpha}\wedge\mathbf{l} = 0.$$

This is the second condition of consistency, and it follows that

either
$$1 = c(l) \sum_p (l_p/\bar{c}_p),$$

or
$$\boldsymbol{\alpha}\wedge\mathbf{l} = 0.$$

The first alternative must be rejected, for we can always find a direction $\mathbf{l}$, such that $\sum\limits_p (l_p/\bar{c}_p) = 0$, and the corresponding speed $c(l)$ would then be infinite. Hence the second alternative must be adopted, and we then deduce that

$$\boldsymbol{\alpha} = 0, \quad \text{i.e.} \quad A_k = 0. \tag{4.10}$$

Equation (4.9) then reduces to

$$B^* c^2(l) = A, \tag{4.11}$$

so that $c(l)$ is a constant c, and the speed of light is the same in all directions in the system S.

5. The Transformation is Lorentzian

It is now easy to show that the transformation is Lorentzian. From (4.4), (4.8), (4.10), (4.11) it follows that

$$\left.\begin{aligned} \sum_j \beta_{ji}\beta_{jk} - \gamma^2\beta_{4i}\beta_{4k} &= B^*\delta_{ki} \\ \sum_j \beta_{j4}\beta_{jk} - \gamma^2\beta_{44}\beta_{4k} &= 0 \\ \text{and} \qquad \sum \beta_{j4}\beta_{j4} - \gamma^2\beta_{44}\beta_{44} &= -B^*c^2 \end{aligned}\right\}. \tag{5.1}$$

Hence, if

$$\xi_\mu = \sum_\nu \beta_{\mu\nu} x_\nu + \beta_\mu \tag{5.2}$$

and

$$\eta_\mu = \sum_\nu \beta_{\mu\nu} y_\nu + \beta_\mu,$$

then

$$\begin{aligned} &\sum_j (\xi_j - \eta_j)^2 - \gamma^2(\xi_4 - \eta_4)^2 \\ &\quad = \sum_{j,\mu,\nu} \beta_{j\mu}\beta_{j\nu}(x_\mu - y_\mu)(x_\nu - y_\nu) - \gamma^2 \sum_{\mu,\nu} \beta_{4\mu}\beta_{4\nu}(x_\mu - y_\mu)(x_\nu - y_\nu) \\ &\quad = B^*\Big\{\sum_j (x_j - y_j)^2 - c^2(x_4 - y_4)^2\Big\}. \end{aligned} \tag{5.3}$$

This relation is sufficient to show that the transformation is a generalized Lorentz transformation, i.e. it is the product of

(i) translations $\xi_\mu = x_\mu + \beta_\mu$;

(ii) spatial rotations, for which $\xi_4 = x_4$, $\sum \xi_j^2 = \sum x_j^2$;

(iii) special Lorentz transformations in (ξ_1, ξ_4) and (x_1, x_4), etc.;

(iv) magnifications $\xi_j = m x_j$, $\xi_4 = n x_4$.

By means of the orthogonality relations (5.1) we can solve the transformation equations (5.2) for x_ν, obtaining

$$\left.\begin{aligned} B^* x_i &= \sum_j \xi_j \beta_{ji} - \gamma^2 \xi_4 \beta_{4i} + \text{constant} \\ \text{and} \qquad -B^* c^2 x_4 &= \sum_j \xi_j \beta_{j4} - \gamma^2 \xi_4 \beta_{44} + \text{constant} \end{aligned}\right\}. \tag{5.4}$$

If we now substitute these values of x_ν in (5.3), we obtain a second set of orthogonality relations, one of which is

$$\sum \beta_{4i}^2 - c^{-2}\beta_{44}^2 = -B^*/\gamma^2. \tag{5.5}$$

We can now compare the speed of Σ_0 ($\xi_j = 0$) in the system S with the speed of S_0 ($x_j = 0$) in the system Σ. The coordinates of S_0 in the system Σ are

$$\xi_\mu = \beta_{\mu 4} x_4 + \beta_\mu,$$

and hence its speed is

$$u_\Sigma = \left\{\sum_j \left(\frac{d\xi_j}{d\xi_4}\right)^2\right\}^{\frac{1}{2}} = \left\{\sum_j \beta_{j4}^2\right\}^{\frac{1}{2}} / |\beta_{44}|.$$

Therefore, by (5.1), $\qquad u_\Sigma^2 = \gamma^2 - B^* c^2/\beta_{44}^2. \qquad$ (5.6)

The coordinates of Σ_0 in the system S are given by

$$B^* x_i = \gamma^2 \xi_4 \beta_{4i} + \text{constant}$$

and

$$B^* c^2 x_4 = \gamma^2 \xi_4 \beta_{44} + \text{constant},$$

by (5.4). Hence the speed of Σ_0 in S is

$$u_S = \left\{\sum_j \left(\frac{dx_i}{dx_4}\right)^2\right\}^{\frac{1}{2}} = c^2\left\{\sum_j \beta_{4i}^2\right\}^{\frac{1}{2}} / |\beta_{44}|.$$

Therefore, by (5.5),

$$u_S^2 = c^2 - (B^* c^4)/(\gamma^2 \beta_{44}^2). \tag{5.7}$$

It now follows from (5.6) and (5.7) that

$$u_\Sigma^2 : u_S^2 = \gamma^2 : c^2. \tag{5.8}$$

This result shows that, if the units of measurement in S and Σ are chosen so that $u_S = u_\Sigma$, then the speed of light will be the same in both systems.

6. The Transformation in General Relativity

The assumption which we have hitherto made regarding the equivalence of a pair of observers is that the optical situation which appears as a parallel beam of light to one observer will appear as a parallel beam of light to the other observer. This assumption was given the dual interpretation that each observer would regard the beam of light *either* as a system of plane waves advancing normal to themselves *or* as a system of photons moving along rectilinear rays in the same direction as the waves and with the same speed,

which might vary with the direction in the system of reference of one of the observers. From the standpoint of the physicist this assumption is subject to the tacit understanding that it must refer only to the optical situation in the immediate neighbourhood of the two observers, who must accordingly be neighbouring observers. Measurements made by an actual observer are made *at* the observer and have immediate reference only to neighbouring events. The explicit recognition of this condition enables us to dispense with our original assumption that there is agreement between our observers as to which optical situations appear as parallel beams of light.

If we confine ourselves, as we must, to events in the immediate neighbourhood of an observer, *any* wave front, in general, will appear to be plane, and the track of *any* photon will appear to be straight. The preceding analysis can therefore be taken over into the general theory of relativity with the understanding that the coordinates (x_μ) and (ξ_μ) are to be replaced by the differentials (dx_μ) and $(d\xi_\mu)$. Equations (2.1) to (2.4) will then refer to the tangent plane to a wave front and to the tangent line to a ray.

There will be a consequential simplification in the argument, for the transformation from $(d\xi_\mu)$ to (dx_μ) is necessarily *linear*. We can therefore omit § 3 and proceed at once to the proof in § 4 that, if the speed of light is the same in all directions for any one observer, then the same must be true for all neighbouring observers. The argument of § 5 then shows that

$$\sum_j dx_j^2 - c^2\,dx_4^2 = \sum_j d\xi_j^2 - c^2\,d\xi_4^2,$$

it being understood that (dx_μ) and $(d\xi_\mu)$ are Galilean coordinates. We thus demonstrate the invariance of the metric in the four-dimensional manifold of general relativity.

Finally, I wish to thank Professor E. A. Milne, whose criticisms of the original draft of this paper stimulated me to discuss the problem in a much more general and fundamental manner.

MULTIPOLAR AND MULTIGLOBULAR COORDINATES

By E. H. NEVILLE (*Reading*)

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THE object of this paper is to establish the correlation of circles in a plane with points in space, and the theory of tetracyclic and pentaspherical coordinates, on a direct elementary basis instead of by means of the cartesian equations of circles and spheres. The general argument is presented in a form independent of the number of dimensions, but in § 3 an application is made to a theorem on conics in a plane. I have not delayed to discuss the modifications which allow a globe* to degenerate into a prime, since these modifications are the same however the subject is approached.

1. The specification of a point P in a flat space S_n of n dimensions by its distances from n base points is necessarily ambiguous, for, if Q is the reflection of P in the flat space S_{n-1} which contains the base points, the n distances of Q are the same as the n distances of P; it is only if P is in the prime containing the base points that Q coincides with P and the ambiguity disappears. But to say that the ambiguity does disappear in this case asserts that, in a space of $n-1$ dimensions, specification by distances from n points which are not contained in a flat space of $n-2$ dimensions is complete. It is not necessary to introduce directions of measurement along the several radii, and for coordinates we take the squares of the distances.

We denote the *multipolar coordinates*, so defined, by $\lambda_1, \lambda_2, \ldots, \lambda_n$, or, if we wish our equations to be homogeneous, by $\lambda_1/\lambda_{n+1}, \lambda_2/\lambda_{n+1}, \ldots, \lambda_n/\lambda_{n+1}$. The system is redundant, and the permanent relation is familiar, being nothing but the Cayley-Sylvester identity between the mutual distances of $n+1$ points in space of $n-1$ dimensions. If

* The unappropriated word 'globe' is a useful addition to the vocabulary of geometrical terms without dimensional implication. In one dimension the globe is a point-pair; in two it is a circle. If we are to take an adjective also from everyday speech, we cannot evade the inconsistencies of language; 'global' exists, as do half a dozen other derivatives, in the sense that the compilers of dictionaries have found them in print, but it is 'globular' which, as the *O.E.D.* says, 'although etymologically related to *globule*, is commonly employed in senses corresponding to those of *globe*', and it is 'globular' which is applied by chemists to some of their largest molecules and by astronomers to stellar configurations.

the base points are $A_1, A_2,..., A_n$, and if a_{rs} denotes $A_r A_s^2$, the identity is

$$\begin{vmatrix} 0 & 1 & 1 & 1 & . & . & . & 1 & 1 \\ 1 & 0 & a_{12} & a_{13} & . & . & . & a_{1n} & \lambda_1 \\ 1 & a_{21} & 0 & a_{23} & . & . & . & a_{2n} & \lambda_2 \\ . & . & . & . & . & . & . & . & . \\ 1 & a_{n1} & a_{n2} & a_{n3} & . & . & . & 0 & \lambda_n \\ 1 & \lambda_1 & \lambda_2 & \lambda_3 & . & . & . & \lambda_n & 0 \end{vmatrix} = 0, \tag{1.1}$$

or in homogeneous form, with the columns and rows in the order that is then more natural,

$$\begin{vmatrix} 0 & a_{12} & a_{13} & . & . & . & a_{1n} & 1 & \lambda_1 \\ a_{21} & 0 & a_{23} & . & . & . & a_{2n} & 1 & \lambda_2 \\ . & . & . & . & . & . & . & . & . \\ a_{n1} & a_{n2} & a_{n3} & . & . & . & 0 & 1 & \lambda_n \\ 1 & 1 & 1 & . & . & . & 1 & 0 & \lambda_{n+1} \\ \lambda_1 & \lambda_2 & \lambda_3 & . & . & . & \lambda_n & \lambda_{n+1} & 0 \end{vmatrix} = 0. \tag{1.2}$$

The permanent relation between the homogeneous multipolar coordinates (λ_r) in space of $n-1$ dimensions is the condition that the prime $\lambda_r x^r = 0$ in n dimensions touches the quadric $a_{rs} x^r x^s = 0$, where

$$a_{rs} = A_r A_s^2, \quad a_{r,n+1} = a_{n+1,s} = 1, \quad a_{n+1,n+1} = 0 \qquad (r,s = 1,2,..., n). \tag{1.3}$$

In other words, the use of multipolar coordinates sets up a one-one correspondence between the points of S_{n-1} and the tangent primes to a quadric in S_n.

2. In S_n, a linear relation $u^r\lambda_r = 0$ is susceptible of four interpretations. Most directly, the relation expresses that the prime (λ_r) passes through the point (u^r), which, if the prime touches the base quadric K, is a point outside the quadric. The aggregate of points in S_{n-1} for which the multipolar coordinates satisfy the relation $u^r\lambda_r = 0$ can therefore be correlated with the aggregate of tangent primes to K through (u^r), with the point (u^r) itself, with the prime $(a_{rs} u^s)$ which is the polar of (u^r) for K, or with the quadric in $n-1$ dimensions which in the section of K by $(a_{rs} u^s)$.

In S_{n-1}, if $\bar{u} \equiv u^1+u^2+...+u^n \neq 0$, and if G_u is the mean centre of loads $u^1, u^2,..., u^n$ at $A_1, A_2,..., A_n$, then identically for any point P,

$$u^1.A_1P^2+u^2.A_2P^2+...+u^n.A_nP^2 = (u^1+u^2+...+u^n)G_uP^2+ \\ +(u^1.A_1G_u^2+u^2.A_2G_u^2+...+u^n.A_nG_u^2). \tag{2.1}$$

Hence, if $\bar{u} \neq 0$, the linear relation $u^r\lambda_r = 0$ is equivalent in S_{n-1} to

$$G_u P^2 = -u^r\gamma_r/\bar{u}\gamma_{n+1}, \tag{2.2}$$

where $\gamma_1 : \gamma_2 : ... : \gamma_{n+1}$ are the multipolar coordinate-ratios of the mean centre G_u.

An alternative analysis enables us to deal with the exceptional case. With any origin O in S_{n-1},

$$A_r P^2 = OP^2 - 2(OP \, . \, OA_r) + OA_r^2,$$

where $(OP \, . \, OA_r)$ is the scalar product of the vectors of the steps OP, OA_r; hence, if OU is the step from O with the vector

$$u^1 . OA_1 + u^2 . OA_2 + ... + u^n . OA_n,$$

and if D is the projection of P on the line OU, then

$$\begin{aligned} &u^1 . A_1 P^2 + u^2 . A_2 P^2 + ... + u^n . A_n P^2 \\ &\quad = \bar{u} . OP^2 - 2OU . OD + (u^1 . OA_1^2 + u^2 . OA_2^2 + ... + u^n . OA_n^2) \end{aligned} \tag{2.3}$$

and the relation $u^r\lambda_r = 0$ is equivalent to

$$\bar{u}\omega_{n+1} . OP^2 - 2\omega_{n+1} . OU . OD + u^r\omega_r = 0, \tag{2.4}$$

where $\omega_1 : \omega_2 : ... : \omega_{n+1}$ are the coordinates of O. If $\bar{u} \neq 0$, OU is $\bar{u} . OG_u$, and, if O is taken at G_u, the result already found is recovered. If $\bar{u} = 0$, the vector $u^1 . OA_1 + u^2 . OA_2 + ... + u^n . OA_n$ is independent of the position of O in S_{n-1}, since for any two points O, Q in any case

$$\begin{aligned} &(u^1 . OA_1 + u^2 . OA_2 + ... + u^n . OA_n) \\ &\qquad\qquad = \bar{u} . OQ + (u^1 . QA_1 + u^2 . QA_2 + ... + u^n . QA_n); \end{aligned}$$

hence this vector is not zero unless u^1, u^2,..., u^n are all zero, for, if $u^r \neq 0$ and s is distinct from r, the relation

$$u^1 . A_s A_1 + u^2 . A_s A_2 + ... + u^n . A_s A_n = 0$$

would imply that the base points A_1, A_2,..., A_n were contained in a flat space of $n-2$ dimensions; it follows that, if $\bar{u} = 0$, and if u^1, u^2,..., u^n are not all zero, the relation $u^r\lambda_r = 0$ is equivalent to

$$OD = u^r\omega_r/2\omega_{n+1} . OU \tag{2.5}$$

and expresses that D is a fixed point, that is, that P is in the S_{n-2} through D orthogonal to OU. Lastly, if u^1, u^2,..., u^n are all zero, the linear relation $u^r\lambda_r = 0$ becomes $\lambda_{n+1} = 0$ and expresses that P is at infinity.

(2.6) *The linear equation in homogeneous multipolar coordinates is the general equation of a globe, and includes as particular cases the equation of a prime and the condition satisfied by points at infinity. By means of multipolar coordinates, the globes in S_{n-1} are correlated with the points of S_n outside a quadric K, and with the primes of S_n which cut K.*

That it is the points outside K which correspond to globes in S_{n-1} can be seen analytically by a transformation of the sum $u^r\gamma_r$. In the identity (2.1), let P be taken at the base point A_s; we have then

$$a_{1s}u^1+a_{2s}u^2+...+a_{ns}u^n$$
$$= \bar{u}.G_u A_s^2+(u^1.A_1 G_u^2+u^2.A_2 G_u^2+...+u^n.A_n G_u^2),$$

and this identity, with the addition of the arbitrary constant u^{n+1} to each side, can be written

$$a_{rs}u^r = (\bar{u}\gamma_s+u^r\gamma_r)/\gamma_{n+1} \quad (s = 1, 2,..., n); \tag{2.7}$$

also identically $\quad a_{r,n+1}u^r = \bar{u} = \bar{u}\gamma_{n+1}/\gamma_{n+1}. \quad$ (2.8)

Multiplying the $n+1$ identities by u^1, u^2,..., u^{n+1} and adding, we have on the left the sum $a_{rs}u^r u^s$, in which, since $a_{rr} = 0$ for all values of r, each term occurs twice, while on the right the product $\bar{u}.u^r\gamma_r/\gamma_{n+1}$ occurs in two forms. Thus

$$\bar{u}.u^r\gamma_r/\gamma_{n+1} = \tfrac{1}{2}a_{rs}u^r u^s,$$

and, if $\bar{u} \neq 0$, the equation $u^r\lambda_r = 0$ is equivalent to

$$G_u P^2 = -\tfrac{1}{2}a_{rs}u^r u^s/\bar{u}^2. \tag{2.9}$$

It is only if the point (u^n) is outside the quadric $a_{rs}x^r x^s = 0$ that $a_{rs}u^r u^s$ is negative and that there are real points in S_{n-1} whose multipolar coordinates satisfy the equation $u^r\lambda_r = 0$.

The condition $\bar{u} = 0$ is the equation in S_n of the tangent prime T_{n-1} to the base quadric at the reference vertex X_{n+1}. To say that a prime in S_{n-1} is represented by an equation $u^r\lambda_r = 0$ in which $\bar{u} = 0$ is therefore to correlate the primes of S_{n-1} with the points of this tangent prime. If the correlation is confined to these two primes, a point in S_{n-1} is associated not with a prime (λ_r) in S_n but with the prime section of T_{n-1} by (λ_r). The correlation is a linear correlation of the points in one space with the primes in another space of the same number of dimensions.

3. As an example of a transformation from a theorem of a commonplace kind in geometry of three dimensions to an unfamiliar theorem

in plane geometry, let A, B, C be the points of reference in a plane, and let β, γ be two conics, of which one has foci A, C, and the other has foci A, B. If the homogeneous tripolar coordinates are $(\lambda, \mu, \nu, \varpi)$, the equation of γ is of the form $\sqrt{\lambda}+\sqrt{\mu} = \sqrt{h}.\sqrt{\varpi}$. In space, this is the condition for the plane $\lambda x+\mu y+\nu z+\varpi t = 0$ to touch a conic Γ in the plane $z = 0$, with the equation $hxy+(x+y)t = 0$, and the points of the conic γ are represented* in space by planes which touch both the base quadric K and the conic Γ. Similarly, the tripolar equation of β is $\sqrt{\lambda}+\sqrt{\nu} = \sqrt{g}.\sqrt{\varpi}$, and the points of β are represented in space by planes which touch both K and the conic B which is in the plane $y = 0$ and has the equation $gzx+(z+x)t = 0$. Hence the points of intersection of β and γ are represented by planes which touch the quadric K and the two conics B, Γ. Now the conics B, Γ are not unrelated; they intersect in the points T and X. Hence the developable circumscribing these conics degenerates into a pair of quadric cones, Ξ', Ξ'', and a tangent plane to K which touches B and Γ is a common tangent plane to K and one or other of these cones. But B and Γ are specialized still further in relation to K. The tangents to B and Γ at T are the intersections of the planes of the conics by the plane $x+y+z = 0$, that is, by the tangent plane to K at T. It follows that the vertices U', U'' of Ξ', Ξ'' are in this tangent plane, and that, if E', E'' are the tangent cones from U', U'' to K, then E' touches Ξ' along the line $U'T$, and E'' touches Ξ'' along the line $U''T$. Thus the tangent plane to K at T, which has nothing to do with the conics β, γ, accounts for two of the common tangent planes to the cones E', Ξ', and for two of the common tangent planes to the cones E'', Ξ''; the other two common tangent planes to E', Ξ' and the other two common tangent planes to E'', Ξ'' represent the four points of intersection of β and γ. Since the line of intersection of any two tangent planes to Ξ' passes through the vertex U', and the line of intersection of any two tangent planes to

* The distinction between $\sqrt{\lambda}+\sqrt{\mu}$ and $\sqrt{\lambda}-\sqrt{\mu}$ disappears in the rationalization, but there are not two conics $\sqrt{\lambda}\pm\sqrt{\mu} = \sqrt{h}.\sqrt{\varpi}$, for the negative sign is impossible in the plane if h is greater than AB^2 and the positive sign is impossible if h is less than AB^2. There are points of Γ corresponding to each sign; Γ passes through the two vertices X, Y of the triangle of reference in the plane XYT, and each sign is associated with one of the two arcs into which these vertices divide the conic, but Γ crosses the base quadric K at these points, and the real tangent planes common to Γ and K belong to one of the arcs only.

Ξ'' passes through the vertex U'', and U', U'' are points in the tangent plane to K at T, it follows that

(3.1) *Two of the common chords of the conics β, γ are represented in space by the vertices of the two cones through the conics* B, Γ.

That this theorem should discriminate between two of the six common chords and the other four is not hard to understand, for it is an immediate corollary of the definition of a conic by means of focus and directrix that, if two conics have a common focus, two of their common chords pass through the intersection of the directrices associated with this focus; the common chords of β and γ represented by U' and U'' pass through the intersection of the directrices associated with A.

We now introduce a conic α whose foci are B and C, represented in space by a conic A in the plane $x = 0$ with the equation

$$fyz+(y+z)t = 0.$$

There are common chords of γ and α represented by the vertices V', V'' of the cones through Γ and A, and there are common chords of α and β represented by the vertices W', W'' of the cones through A and B. The conics A, B, Γ are three sections of the one quadric

$$fyz+gzx+hxy+(x+y+z)t = 0.$$

Hence* the pairs of vertices U', U''; V', V''; W', W'' are the pairs of vertices of a quadrilateral, and the corresponding pairs of lines in the plane ABC are the pairs of sides of a quadrangle.

(3.2) *If, in a plane ABC, α, β, γ are conics with foci B, C, with foci C, A, and with foci A, B, there are a pair of common chords of β and γ, a pair of common chords of γ and α, and a pair of common chords of α and β, which are the pairs of sides of a quadrangle.*

If two ellipses have a common focus, they cannot have more than two real points of intersection, and it was proved by J. S. Turner that, if three ellipses share their foci in the way described in (3.2), and if each pair of ellipses has a chord of visible intersection, then the three chords of visible intersection are concurrent. An

* The six vertices are in the polar plane of the point of intersection of the planes of the three conics, and the collinearities required are demonstrable by applications of Pascal's theorem to the intersection of the quadric by this plane. This proof fails in the very case with which we are concerned, since the Pascal figures degenerate, but the cones do not degenerate or coalesce, and the theorem remains true.

elementary proof of Turner's result and of (3.2), and a discussion of the relation between them, have been given elsewhere;* actually the proof of (3.2) given above was found before the elementary proof, but the assumption that the interpretation put on the Cayley-Sylvester relation is so obvious as to be common knowledge seems to have been mistaken. The ideas elaborated in the present paper are implicit in the application of the interpretation to the simple case of the geometry of point-pairs on a line, which is given in detail in another paper.†

4. Since the mean centre G_u which is the centre of the globe $u^r \lambda_r = 0$ is determined by the ratios $u^1 : u^2 : \ldots : u^n$, different globes in S_{n-1} which have the same centre G_u are represented in S_n by different points on the line

$$\frac{x^1}{u^1} = \frac{x^2}{u^2} = \ldots = \frac{x^n}{u^n}.$$

But, since this line passes through a vertex of reference which is on the base quadric K, the line is partly inside and partly outside the quadric, and the points inside the quadric, though they seem to be associated with the point G_u as closely as the points outside the quadric, do not correspond to globes in S_{n-1}. To extend the correspondence by introducing spheres of imaginary radius, that is, by considering the spaces S_{n-1} and S_n as essentially complex, is to change the subject too drastically. What the correspondence suggests is rather that the global‡ geometry of S_{n-1} is a geometry in which the element is a point G of S_{n-1} associated with a radial measure Γ which may be positive, zero, or negative. If Γ is not negative, the element $[G, \Gamma]$ specifies the actual globe which is the aggregate of the points P for which $GP^2 = \Gamma$. A theorem in which no negative measures occur can be read as a theorem concerning actual globes; the interest of a theorem in which there are measures of both signs must be intrinsic.

* 'A focus-sharing set of three conics': *Math. Gazette*, 20 (1936), 182.

† 'Bipolar and trigeminal coordinates on a line': *J. of the Indian Math. Soc.* (Ser. 2) 2 (1937), 173. The paper was taken as read before the London Math. Soc., 23 April 1936.

‡ We have to distinguish between the geometry of configurations of globes and the geometry of figures in a globe; if analogy with the established meaning of spherical geometry and spherical trigonometry is to be preserved, it is the latter that must be called globular geometry, and, unless we now admit a second adjective, the former must be the geometry of globes.

It is now a matter of definition that the linear function $u^r\lambda_r$, which cannot necessarily be equated to zero, is associated on the one hand in S_n with the point (u^r) or the prime $(a_{rs}u^s)$, and on the other hand in S_{n-1} with the point G_u and the measure Γ_u which are such that

$$u^r\lambda_r = \bar{u}\lambda_{n+1}(G_u P^2 - \Gamma_u) \tag{4.1}$$

for all positions of P. The formulae connecting the representation in S_n with the globular element in S_{n-1} are implicit in this identity. If P is the base point A_r, λ_s/λ_{n+1} is by definition a_{sr} and $G_u P^2$ is γ_r/γ_{n+1}; thus

$$a_{sr}u^s\gamma_{n+1} = \bar{u}(\gamma_r - \gamma_{n+1}\Gamma_u) \quad (r = 1, 2,..., n). \tag{4.2}$$

Also, if P is G_u, λ_r/λ_{n+1} is γ_r/γ_{n+1} and $G_u P$ is zero, whence

$$u^r\gamma_r = -\bar{u}\gamma_{n+1}\Gamma_u, \tag{4.3}$$

while, if r is $n+1$, $a_{sr}u^s$ is $\bar{u}$, and we may write

$$a_{s,n+1}u^s\gamma_{n+1} = \bar{u}\gamma_{n+1}. \tag{4.4}$$

From (4.2) and (4.3) we have, as in (2.9),

$$a_{rs}u^ru^s = -2\bar{u}^2\Gamma_u. \tag{4.5}$$

Supposing (u^r) to be given, we have the radial measure Γ_u from (4.5), and then the homogeneous multipolar coordinates $\gamma_1, \gamma_2,..., \gamma_{n+1}$ from (4.2). If $[G_u, \Gamma_u]$ is given, (4.2) and (4.4) express the ratios of the $n+1$ linear combinations $a_{sr}u^s$ to $\bar{u}$, and the ratios $u^1 : u^2 : ... : u^{n+1}$ may be inferred. But there is a better way of regarding the formulae: as we have said, $(a_{rs}u^s)$ is the prime in S_n which is the polar of (u^r) for the base quadric, and we may say simply that

(4.6) *The globular element* $[G_u, \Gamma_u]$ *in* S_{n-1} *is represented in* S_n *by the prime whose coordinates are*

$$\gamma_1 - \gamma_{n+1}\Gamma_u,\ \gamma_2 - \gamma_{n+1}\Gamma_u, ..., \gamma_n - \gamma_{n+1}\Gamma_u,\ \gamma_{n+1}.$$

This representation brings us back to our original correlation of a point with a prime, for, if Γ_u is zero, the coordinates of the prime become the multipolar coordinates of the point G_u. To find the globular element represented by a given prime (η_r), we remark that, if Γ_η is the radial measure, the multipolar coordinates of the centre G_η are

$$\eta_1 + \eta_{n+1}\Gamma_\eta,\ \eta_2 + \eta_{n+1}\Gamma_\eta, ..., \eta_n + \eta_{n+1}\Gamma_\eta,\ \eta_{n+1}.$$

Substituting these values of the coordinates in the permanent relation (1.2), we have

$$2\eta_{n+1}^2\,\Gamma_\eta \begin{vmatrix} 0 & a_{12} & a_{13} & . & . & . & a_{1n} & 1 \\ a_{21} & 0 & a_{23} & . & . & . & a_{2n} & 1 \\ . & . & . & . & . & . & . & . \\ a_{n1} & a_{n2} & a_{n3} & . & . & . & 0 & 1 \\ 1 & 1 & 1 & . & . & . & 1 & 0 \end{vmatrix}$$

$$= \begin{vmatrix} 0 & a_{12} & a_{13} & . & . & . & a_{1n} & 1 & \eta_1 \\ a_{21} & 0 & a_{23} & . & . & . & a_{2n} & 1 & \eta_2 \\ . & . & . & . & . & . & . & . & . \\ a_{n1} & a_{n2} & a_{n3} & . & . & . & 0 & 1 & \eta_n \\ 1 & 1 & 1 & . & . & . & 1 & 0 & \eta_{n+1} \\ \eta_1 & \eta_2 & \eta_3 & . & . & . & \eta_n & \eta_{n+1} & 0 \end{vmatrix} \qquad (4.7)$$

and, when Γ_η has been found from this formula, the coordinates of G_η follow at once.

5. If in the identity (4.1) we take for P the centre G_ζ of the globe corresponding to the prime (ζ_r), we have, from the values of the multipolar coordinates just found,

$$u^r\zeta_r + \bar{u}\zeta_{n+1}\Gamma_\zeta = \bar{u}\zeta_{n+1}(G_u\,G_\zeta^2 - \Gamma_u),$$

that is,
$$u^r\zeta_r = \bar{u}\zeta_{n+1}(G_u\,G_\zeta^2 - \Gamma_u - \Gamma_\zeta). \qquad (5.1)$$

The function $G_u\,G_\zeta^2 - \Gamma_u - \Gamma_\zeta$, which depends only on the two globes and not on the systems of reference, is the mutual power of the globes, and we may say simply that

(5.2) *The mutual power of the globes* (u^r), (ζ_r) *is* $u^r\zeta_r/\bar{u}\zeta_{n+1}$.

If the second globe is defined by the point (v^r), we have

$$\zeta_r/\zeta_{n+1} = a_{rs}v^s/\bar{v},$$

for all values of r, and therefore

(5.3) *The mutual power of the globes* (u^r), (v^r) *is* $a_{rs}u^rv^s/\bar{u}\bar{v}$.

If the first globe is defined by the prime (η_r), the substitution takes the reverse form $u^r/\bar{u} = a^{rs}\eta_s/\eta_{n+1}$, and

(5.4) *The mutual power of the globes* (η_r), (ζ_r) *is* $a^{rs}\eta_r\,\zeta_s/\eta_{n+1}\,\zeta_{n+1}$.

The coefficients a^{rs} are, of course, formed from the complete set of coefficients of the base quadric as defined in (1.3), not from the squares $A_r\,A_s^2$ alone. Since the self-power of the globe $[G, \Gamma]$ is -2Γ, the evaluations of Γ_u and Γ_η in (4.5) and (4.7) are particular cases of (5.3) and (5.4).

6. Correspondence with the points of S_n implies that globes in S_{n-1} fall into families with the same formal relations of inclusion and dependence as the flat sub-spaces of S_n, and (5.2) enables us to describe the constitution of these families in terms of mutual relations between globes. In S_n a prime consists of all the points which are conjugate for the quadric K to some one point, the pole of the particular prime. An S_{n-p} is the space common to p linearly independent primes, and consists of all the points which are conjugate simultaneously to their poles. A point which is conjugate to p linearly independent points is conjugate to every point in the S_{p-1} which these points determine, and with reference to the quadric the flat sub-spaces of S_n are in conjugate pairs, complementary in dimensions; S_q, S_r are conjugate if every point in one is conjugate to every point in the other; if also $q+r = n-1$, each sub-space is determined uniquely by the other, and the complementary pair is determined by any set of linearly independent points which determines one of the two sub-spaces. In particular, the aggregate of points conjugate to two points V_1, V_2 constitutes a sub-space S_{n-2}, the intersection of the polars of V_1, V_2, and the points which are conjugate to all the points in this sub-space compose the line $V_1 V_2$. Having recognized the line in this way, we can build the flat spaces of higher dimensions from the line instead of regarding each space as generated by the relation of conjugacy to a set of points in the complementary space.

It follows at once from (5.2) that the condition for the points in S_n which represent the globes $[G, \Gamma]$, $[H, \Delta]$ to be conjugate for the base quadric is that the mutual power of the globes is zero. If the globes are actual and if $n > 2$, this condition, $GH^2 = \Gamma+\Delta$, expresses that the globes cut at right angles; we can therefore say consistently that two globes are orthogonal if their mutual power is zero. (If $n = 2$, the space S_{n-1} is a line, and the globe, if actual, is a pair of points; the power of two pairs vanishes if the pairs are harmonic to each other.)

It is not worth while to do more than translate one or two theorems. If A, B are any two globes in a flat space of $n-1$ dimensions, the globes orthogonal to them both compose an $(n-2)$-fold of globes, and the globes orthogonal to every member of this $(n-2)$-fold compose the linear set or pencil of globes determined by A and B. Two globes determine one and only one pencil, and, if C and D belong to the pencil determined by A and B, then A and B belong

to the pencil determined by C and D. If A, B, C do not belong to one pencil, any pencil which includes a globe belonging to the pencil AB and a globe belonging to the pencil AC includes also a globe belonging to the pencil BC; the aggregate of members of pencils which intersect two intersecting pencils AB, AC is the global plane ABC.

An alternative definition of the linear set of globes is implicit in (5.2). If (u^r), (v^r), (w^r) are collinear in S_n, there are constants h', k' such that $w^r = h'u^r + k'v^r$ for all values of r. Hence, if (ζ_r) is any globe,

$$\Pi_{w\zeta} = \frac{h'.u^r\zeta_r + k'.v^r\zeta_r}{(h'\bar{u}+k'\bar{v})\zeta_{n+1}} = \frac{h'\bar{u}.\Pi_{u\zeta}+k'\bar{v}.\Pi_{v\zeta}}{h'\bar{u}+k'\bar{v}}.$$

That is, there are constants h, k, independent of the globe (ζ_r), such that

$$\Pi_{w\zeta} = \frac{h\Pi_{u\zeta}+k\Pi_{v\zeta}}{h+k}. \tag{6.1}$$

As a point aggregate, the globe (w^r) consists of the centres of null globes orthogonal to (w^r), and, if the globe ζ is the null globe $[P, 0]$, the powers $\Pi_{u\zeta}$, $\Pi_{v\zeta}$ become the powers $G_u P^2 - \Gamma_u$, $G_v P^2 - \Gamma_v$ of the point P for the globes $[G_u, \Gamma_u]$, $[G_v, \Gamma_v]$.

(6.2) *An actual globe collinear with two globes* $[G, \Gamma]$, $[H, \Delta]$ *is the aggregate of points* P *for which the powers* $GP^2 - \Gamma$, $HP^2 - \Delta$ *satisfy a homogeneous linear relation.*

More generally,

(6.3) *An actual globe belonging to the linear complex determined by* p *globes* $[G_1, \Gamma_1]$, $[G_2, \Gamma_2]$,..., $[G_p, \Gamma_p]$ *is the aggregate of points satisfying an equation*

$$k_1(G_1 P^2 - \Gamma_1) + k_2(G_2 P^2 - \Gamma_2) + ... + k_p(G_p P^2 - \Gamma_p) = 0.$$

We escape from the restriction to actual globes as before by replacing the equation by a functional equivalence:

(6.4) *The globe* $[G, \Gamma]$ *belongs to the complex determined by* p *globes* $[G_1, \Gamma_1]$, $[G_2, \Gamma_2]$,..., $[G_p, \Gamma_p]$ *if there are constants* k_1, k_2,..., k_p, $\bar{k}$ *such that*

$$k_1(G_1 P^2 - \Gamma_1) + k_2(G_2 P^2 - \Gamma_2) + ... + k_p(G_p P^2 - \Gamma_p) = \bar{k}(GP^2 - \Gamma)$$

identically, for all positions of P.

Transforming the sum $\sum k_m(G_m P^2 - \Gamma_m)$ as before, we conclude that $\bar{k}$ is $\sum k_m$, that G is the mean centre of loads k_m at G_m, for $m = 1, 2,...,$ p, and that Γ is given by

$$-\Gamma = \sum k_m(G_m G^2 - \Gamma_m)/\bar{k}. \tag{6.5}$$

Hence G belongs to the flat sub-space which contains $G_1, G_2, \ldots, G_p$; conversely, if G is any point of this sub-space, the barycentric coordinates of G with reference to $G_1, G_2, \ldots, G_p$ have determinate ratios, and substituted in (6.5), give a unique value of Γ to be associated with G.

(6.6) *The centres of the globes which compose the complex determined by p linearly independent globes form the flat space of dimensions $p-1$ containing the centres of these p globes, and each point of this space is the centre of one and only one member of the complex.*

7. Let a globe $[G, \Gamma]$ in S_{n-1} be represented in S_n by the point O. A point B in S_{n-1}, regarded as a null globe, is represented by a point U on the base quadric, and the line OU cuts the base quadric again in a point V which represents a point C in S_{n-1}. Incorporating the fundamental property of the globes associated with collinear points, we infer that

(7.1) *All the globes which are orthogonal to a given globe and pass through one fixed point pass through a second fixed point also.*

Two points related in this way to a globe are said to be inverse with respect to the globe.

Translating the relation of collinearity differently, we may say that A, B are points such that, for all positions of P,

$$h.AP^2+k.BP^2 \equiv (h+k)(GP^2-\Gamma), \tag{7.2}$$

where h, k are independent of P. As a special case of (6.6), A, B, G are collinear, and then

$$BG.AP^2+GA.BP^2+AB.GP^2 = -BG.GA.AB,$$

that is, $$GB.AP^2+AG.BP^2 = AB(GP^2-GA.GB).$$

The identity (7.2) must be equivalent to this, and therefore

(7.3) *The inverse of A in $[G, \Gamma]$ is the point B in the line GA such that $GA.GB = \Gamma$.*

8. As a locus in S_{n-1}, an actual globe $[H, \Delta]$ is the aggregate of null globes orthogonal to $[H, \Delta]$. It follows that as a locus this globe is represented in S_n by the aggregate of points on the base quadric conjugate to the point representing $[H, \Delta]$. In other words, the points of the actual globe are represented by the points of the section of the base quadric by a prime. If O represents a globe $[G, \Gamma]$, the lines joining O to the points of this section compose a quadric cone and cut the base quadric in the points of a second prime section.

(8.1) *The inverse of an actual globe in any globe, actual or virtual, is an actual globe.*

In S_n, let O, U represent the two globes $[G, \Gamma]$, $[H, \Delta]$, of which the second is actual, and let A be any point on the section of the base quadric by the polar of U. The plane OUA cuts the quadric in a conic, and cuts the polar of U in a line which cuts the conic in A and in another point B; the lines AO, BO cut the conic again in points C, D which represent the inverses in $[G, \Gamma]$ of the points represented by A, B. Let the lines AB, CD cut in E, the lines AD, BC cut in W, and the tangents to the conic at C, D cut in U_γ. The four points O, W, U, U_γ are all on the polar of E for the conic; that is, W and U_γ are on the line OU. Also W is conjugate to O for the conic, and is therefore the intersection of the line OU by the prime which is the polar of O for the quadric. The pair of lines EAB, ECD harmonizes with the pair of lines EO, EW; hence the pair of poles U, U_γ harmonizes with the pair of poles W, O, and, since the points U, W, O are independent of the position of A on the section of the quadric by the polar of U, the point U_γ also is independent of the position of A in this section. But the prime which touches the quadric at C passes through U_γ. Hence U_γ, identifiable as the harmonic of U with respect to O and W on the line OU, is the point which represents the globe inverse to $[H, \Delta]$.

The construction of U_γ from U does not require U to be outside the base quadric, and we can use this construction provisionally to define the inverse of a virtual globe. We proceed to a fundamental theorem from which a definition in terms of constructions in S_{n-1} follows immediately.

Let U, V represent two globes orthogonal to each other. The construction for the points U_γ, V_γ can be completed in the plane OUV. Let the polar of O for the quadric cut the lines OU, OV in W_U, W_V, and let the polar of W_U cut the polar of O in E. Since the four points O, W_U, U, U_γ are collinear, the polars of these points pass through E, and, since U and U_γ harmonize with O and W_U, the lines in which the polars of U and U_γ cut the plane harmonize with the lines in which the polars of O and W_U cut the plane. But the polar of U contains the point V and therefore cuts the plane in the line EV, the polar of W_U cuts the plane in the line EO, and the polar of O cuts the plane in the line EW_U, which is the line EW_V. Thus EV and the line in which the polar of U_γ cuts the plane harmonize with

EO and EW_γ. But V and V_γ harmonize with O and W_γ. Hence V_γ is on the polar of U_γ.

(8.2) *If two globes are orthogonal, their inverses in any globe are orthogonal also.*

It follows that

(8.3) *If two points are inverse in a globe* δ, *their inverses in a globe* γ *are inverse in the globe* δ_γ,

since the inverses of actual globes through A orthogonal to δ are actual globes through A_γ orthogonal to δ_γ.

If A, B are two points on the base quadric, the globes in which the corresponding points in S_{n-1} are inverse to each other are represented by points on the line AB. Since two lines do not in general intersect and in any case cannot have more than one point in common, there is not in general a globe in S_{n-1} for which each of two pairs of points is an inverse pair, and, if there is one such globe, this globe is unique. But we can express (8.3) in the form that

(8.4) *The globe* δ_γ *is a globe such that the inverse in* γ *of every pair of points inverse for* δ *is a pair of points inverse for* δ_γ.

The existence of such a globe being established, the property is a defining property in terms of configurations in S_{n-1}.

A final step can be taken without further reference to S_n:

(8.5) *If two globes are inverse in a globe* δ, *their inverses in a globe* γ *are inverse in the globe* δ_γ.

9. The mutual ratios of the $n+1$ coordinates λ_r are uniquely determined by the mutual ratios of any $n+1$ independent linear combinations $u^s_{(r)}\lambda_s$. If Π_r is the power of the variable globe represented by the prime (λ_s) and the fixed globe represented by the point $(u^s_{(r)})$, and if ϖ_r denotes $u^s_{(r)}\lambda_s$, we have

$$\varpi_r = \bar{u}_{(r)}\Pi_r\lambda_{n+1},$$

where $\bar{u}_{(r)}$ is a constant associated with the fixed globe. Hence a globe in S_{n-1} is uniquely determined by the mutual ratios of its powers with any $n+1$ linearly independent globes. Thus we have the conception of multiglobular coordinates.

(9.1) *The multiglobular coordinates of a globe in space of* $n-1$ *dimensions are specified multiples of the powers of the globe with* $n+1$ *globes of reference; these constitute for the globe an unambiguous*

homogeneous set of coordinates, redundant only in the sense in which a homogeneous set is necessarily redundant.

As a rule we take the powers themselves as the coordinates, the effect of introducing the multipliers being trivial.

Between $n+2$ homogeneous linear combinations of $n+1$ variables there must be a linear relation. Taking for the linear functions the powers of a variable globe ϕ with $n+2$ globes β_r we infer that there are $n+2$ constants $h_1, h_2, \ldots, h_{n+2}$, independent of ϕ, such that

$$h_1\,\Pi(\beta_1\phi)+h_2\,\Pi(\beta_2\phi)+\ldots+h_{n+2}\,\Pi(\beta_{n+2}\phi) = 0.$$

The ratios of the constants can be eliminated if we substitute for ϕ in turn $n+2$ globes γ_s, and therefore, as proved by Frobenius and Darboux,

(9.2) *For any $2n+4$ globes in space of $n-1$ dimensions the determinant*

$$|\Pi(\beta_r\gamma_s)| \quad (r, s = 1, 2, \ldots, n+2)$$

is identically zero.

If β_r and γ_r are both identical, for $r = 1, 2, \ldots, n+1$, with the globe of reference α_r, $\Pi(\beta_r\gamma_s)$ becomes the power α_{rs} which is the typical constant of the coordinate system, and, if β_{n+2}, γ_{n+2} are regarded as variable globes ϕ, ψ, the powers $\Pi(\beta_{n+2}\gamma_r)$, $\Pi(\beta_r\gamma_{n+2})$, for $r = 1, 2, \ldots, n+1$, become the absolute power coordinates $\varpi_r(\phi)$, $\varpi_r(\psi)$ of these globes. We have, therefore, for any two globes,

$$\begin{vmatrix} [\alpha_{rs}] & [\varpi_r(\phi)] \\ [\varpi_s(\psi)] & \Pi(\phi\psi) \end{vmatrix} = 0, \tag{9.3}$$

and for any one globe, of absolute power coordinates (ϖ_r) and radial measure Γ,

$$2|\alpha_{rs}|\Gamma = \begin{vmatrix} [\alpha_{rs}] & [\varpi_r] \\ [\varpi_s] & 0 \end{vmatrix}.$$

10. The last two formulae express the mutual power and the radial measure in terms of powers with globes of reference, but it is the powers themselves and not only their ratios which are involved: the formulae are not in terms of the homogeneous coordinates. The condition for the power or the measure to be zero is, however, homogeneous.

(10.1) *The two globes whose multiglobular coordinates are (ϖ_r'), (ϖ_r'') are orthogonal if*

$$\begin{vmatrix} [\alpha_{rs}] & [\varpi_r'] \\ [\varpi_s''] & 0 \end{vmatrix} = 0.$$

(10.2) *The globe whose multiglobular coordinates are* (ϖ_r) *is a null globe if*

$$\begin{vmatrix} [\alpha_{rs}] & [\varpi_r] \\ [\varpi_s] & 0 \end{vmatrix} = 0.$$

If a globe is null, any set of coordinates of the globe may be regarded as a set of coordinates of the centre; the power of a null globe $[P, 0]$ and another globe γ is nothing but the power of the point P for the globe γ. It follows that a point in S_{n-1} is uniquely determined by the ratios of its powers for $n+1$ globes of reference; the ratios are not independent, for the powers are connected by the homogeneous quadratic relation (10.2). In other words,

(10.3) *A point in* S_{n-1} *is determinable by a homogeneous set of* $n+1$ *multiglobular coordinates; the set is redundant, the coordinates being subject to a permanent quadratic relation.*

A linear equation in multiglobular coordinates, if it can be satisfied, is the equation of an actual globe; a linear function which is not zero for any actual point corresponds nevertheless to a virtual globe. In short, much that has been said of homogeneous multipolar coordinates is true of multiglobular coordinates. Points of S_{n-1} are again associated with tangent primes to a quadric in S_n, or with the points of contact of these primes, but the quadric no longer bears a peculiar relation to the points of reference. We may say that the correlation between S_{n-1} and S_n is unaltered; the change is in the character of the frames of reference: in S_n the points of reference are not restricted to be on the fundamental quadric; in S_{n-1} the globes of reference are not restricted to be null.

In conclusion, we recognize the connexion between multiglobular coordinates and inversion. It follows from the constructions in § 8 that inversion in S_{n-1} corresponds to a homography in S_n in which the pole and the prime of homography are pole and polar for the base quadric, in other words, to a harmonic perspective which transforms the base quadric into itself. In S_{n-1} inversion is therefore a homogeneous linear transformation of multiglobular coordinates in which the permanent quadratic relation is unaltered.

THE GENERAL QUADRIC PRIMAL IN [5] WHICH IS AT THE SAME TIME INSCRIBED AND CIRCUMSCRIBED TO A GIVEN SIMPLEX

By V. C. MORTON (*Aberystwyth*)

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1. By considering the lines of [3] as represented by the points of a quadric in [5], the double-six theorem is equivalent to the result: *If a non-singular quadric in* [5] *contains the six vertices of a simplex but contains no edge,* and also touches five of its primes, then it must touch the sixth.* Algebraically the double-six theorem is thus: *Given a symmetric non-vanishing determinant of order six in which the six principal diagonal elements are zero but no others are zero, then, if the minors of five of these elements are zero, the sixth must be zero.*†

In this note I give a direct proof of the algebraic result by a method which enables the general form of such a determinant to be written down explicitly in terms of ten parameters.

2. Let $|a_{ij}| = \Delta \neq 0$ be a symmetric determinant of order six in which $a_{ii} = 0$ but no $a_{ij} = 0$ $(j \neq i)$, and in which

$$A_{11} = A_{22} = A_{33} = A_{44} = A_{55} = 0,$$

where A_{ij} is the cofactor of a_{ij}. Then by use of suitable three-line determinants of cofactors we obtain the equations

$$A_{12} A_{15} A_{25} = a_{34} a_{36} a_{46} \Delta^2, \tag{1}$$

$$A_{13} A_{15} A_{35} = a_{24} a_{26} a_{46} \Delta^2, \tag{2}$$

$$A_{24} A_{25} A_{45} = a_{13} a_{16} a_{36} \Delta^2, \tag{3}$$

$$A_{34} A_{35} A_{45} = a_{12} a_{16} a_{26} \Delta^2, \tag{4}$$

with six others similarly formed.

From the conditions laid down on $|a_{ij}|$ it follows that no $A_{rs} = 0$ $(r \neq s)$ for $r, s \leqslant 5$.

From equations (1) to (4) we derive the equation

$$\frac{A_{12} A_{34}}{A_{13} A_{24}} = \frac{a_{12} a_{34}}{a_{13} a_{24}}, \tag{5}$$

* If the quadric contains an edge, then two supposedly non-intersecting lines of the double six intersect.

† This result seems to have been first stated by Richmond who, however, gives no direct proof. Vide *Proc. Cambridge Phil. Soc.* 14, 477.

and from other groups we get equations analogous to (5) true for all groups of four suffixes chosen from 1, 2, 3, 4, 5 and for any division into pairs.

Now, since $A_{11} = 0$, there exist five numbers $k_2^{(1)}, k_3^{(1)}, k_4^{(1)}, k_5^{(1)}, k_6^{(1)}$ such that

$$\left.\begin{aligned} a_{23}a_{13}k_3^{(1)}+a_{24}a_{14}k_4^{(1)}+a_{25}a_{15}k_5^{(1)}+a_{26}a_{16}k_6^{(1)} &= 0 \\ a_{23}a_{12}k_2^{(1)}+a_{34}a_{14}k_4^{(1)}+a_{35}a_{15}k_5^{(1)}+a_{36}a_{16}k_6^{(1)} &= 0 \\ a_{24}a_{12}k_2^{(1)}+a_{34}a_{13}k_3^{(1)}+a_{45}a_{15}k_5^{(1)}+a_{46}a_{16}k_6^{(1)} &= 0 \\ a_{25}a_{12}k_2^{(1)}+a_{35}a_{13}k_3^{(1)}+a_{45}a_{14}k_4^{(1)}+a_{56}a_{16}k_6^{(1)} &= 0 \\ a_{26}a_{12}k_2^{(1)}+a_{36}a_{13}k_3^{(1)}+a_{46}a_{14}k_4^{(1)}+a_{56}a_{15}k_5^{(1)} &= 0 \end{aligned}\right\}. \quad (6)$$

We thus obtain

$$\frac{a_{12}k_2^{(1)}}{A_{12}} = \frac{a_{13}k_3^{(1)}}{A_{13}} = \frac{a_{14}k_4^{(1)}}{A_{14}} = \frac{a_{15}k_5^{(1)}}{A_{15}} = \frac{a_{16}k_6^{(1)}}{A_{16}},$$

and no $k^{(1)}$ can be zero except possibly $k_6^{(1)}$.

Similarly, by using the fact that $A_{22} = 0$ it can be shown that numbers $k_1^{(2)}, k_3^{(2)}, k_4^{(2)}, k_5^{(2)}, k_6^{(2)}$ exist which satisfy five equations (7) analogous to (6). Then, as before, we get

$$\frac{a_{12}k_1^{(2)}}{A_{12}} = \frac{a_{23}k_3^{(2)}}{A_{23}} = \frac{a_{24}k_4^{(2)}}{A_{24}} = \frac{a_{25}k_5^{(2)}}{A_{25}} = \frac{a_{26}k_6^{(2)}}{A_{26}},$$

and again no $k^{(2)}$ can be zero except possibly $k_6^{(2)}$.

By use of equations such as (5) we now get

$$k_3^{(1)}/k_3^{(2)} = k_4^{(1)}/k_4^{(2)} = k_5^{(1)}/k_5^{(2)},$$

and, since the coefficients in the first equation of each of (6) and (7) are the same, it follows that each ratio is also equal to $k_6^{(1)}/k_6^{(2)}$. In particular, if $k_6^{(1)}$ is zero, so also is $k_6^{(2)}$. The second set $k^{(2)}$ can now be chosen so that each ratio is unity. The two sets are now

$$\begin{array}{cccccc} & k_2^{(1)}, & k_3^{(1)}, & k_4^{(1)}, & k_5^{(1)}, & k_6^{(1)}, \\ k_1'^{(2)}, & & k_3^{(1)}, & k_4^{(1)}, & k_5^{(1)}, & k_6^{(1)}. \end{array}$$

By similar reasoning, using the set $k^{(1)}$ with $k^{(r)}$ and then $k^{(2)}$ with $k^{(r)}$ $(r = 2, 3, 4, 5)$, we find that there are six numbers $k_1, k_2,..., k_6$ such that, if $x_1, x_2,..., x_6$; $y_1, y_2..., y_6$ are any two rows of $|a_{ij}|$, then

$$k_1x_1y_1+k_2x_2y_2+...+k_6x_6y_6 = 0.$$

Further, k_6 cannot be zero, for if so we should have

$$A_{16} = A_{26} = ... = A_{56} = 0$$

and Δ would be zero. So, finally, using the last row with the first

five rows in turn and eliminating $a_{16}k_1$, $a_{26}k_2, \ldots, a_{56}k_5$, we find that $A_{66} = 0$.

3. By multiplying the rows of the matrix a_{ij} in turn by $\sqrt{k_1}$, $\sqrt{k_2}, \ldots, \sqrt{k_6}$, then the columns in turn by the same numbers and then multiplying a_{ij} by a suitable scalar, we obtain an orthogonal matrix c_{ij} for which all such equations as (1) are still true and in which now $C_{ij} = \pm c_{ij}$.

We now write

$$\left.\begin{aligned} c_{12}c_{13}c_{23} &= \pm c_{45}c_{46}c_{56} = \lambda u_1 \\ c_{12}c_{14}c_{24} &= \pm c_{35}c_{36}c_{56} = -\lambda u_2 \\ c_{12}c_{15}c_{25} &= \pm c_{34}c_{36}c_{46} = -\lambda u_3 \\ c_{13}c_{14}c_{34} &= \pm c_{25}c_{26}c_{56} = -\lambda u_4 \\ c_{13}c_{15}c_{35} &= \pm c_{24}c_{26}c_{46} = \lambda u_5 \\ c_{14}c_{15}c_{45} &= \pm c_{23}c_{26}c_{36} = \lambda u_6 \\ c_{23}c_{24}c_{34} &= \pm c_{15}c_{16}c_{56} = \lambda u_7 \\ c_{23}c_{25}c_{35} &= \pm c_{14}c_{16}c_{46} = -\lambda u_8 \\ c_{24}c_{25}c_{45} &= \pm c_{13}c_{16}c_{36} = \lambda u_9 \\ c_{34}c_{35}c_{45} &= \pm c_{12}c_{16}c_{26} = -\lambda u_{10} \end{aligned}\right\}. \tag{8}$$

If $x_1, \ldots, x_6$; $y_1, \ldots, y_6$ are any two rows of c_{ij}, then $\sum_{r=1}^{6} x_r y_r = 0$. We thus obtain fifteen equations of which a typical member is

$$c_{13}c_{23} + c_{14}c_{24} + c_{15}c_{25} + c_{16}c_{26} = 0,$$

which becomes, by use of the first three and last equations of (8),

$$-u_1 + u_2 + u_3 \pm u_{10} = 0.$$

The fifteen such equations are consistent only when the negative signs* are taken in the second column of (8), and then reduce to the five independent equations†

$$u_6 = -u_1 + u_2 + u_3 + u_4 - u_5, \tag{9}$$

$$u_7 = -u_1 + u_2 + u_4, \tag{10}$$

$$u_8 = u_1 - u_3 + u_5, \tag{11}$$

$$u_9 = u_1 - u_4 + u_5, \tag{12}$$

$$u_{10} = -u_1 + u_2 + u_3. \tag{13}$$

* c_{ij} is thus an improper orthogonal matrix.

† The relations between the numbers u are those between the ten singular solids u_r of Segre's quartic primal $V = 0$ in [4]. Alternative forms of V are given later in this section.

All the fifteen relations are, in fact, given by

$$c_{qr}c_{qs}c_{rs}+c_{pr}c_{ps}c_{rs}+c_{pq}c_{ps}c_{qs}+c_{pq}c_{pr}c_{qr}=0,$$

where p, q, r, s are any four numbers chosen from 1, 2,..., 6.

By multiplying together the left-hand and right-hand sides of the first three and last equations of (8) we get

$$-Cc_{12}^2=\lambda^4 u_1 u_2 u_3 u_{10},$$

where C is the product of all the ten c_{rs} for which $r, s \leqslant 5$. Similarly, we obtain

$$-Cc_{13}^2=\lambda^4 u_1 u_4 u_5 u_9,\quad -Cc_{14}^2=\lambda^4 u_2 u_4 u_6 u_8,\quad -Cc_{15}^2=\lambda^4 u_3 u_5 u_6 u_7,$$

$$-Cc_{23}^2=\lambda^4 u_1 u_6 u_7 u_8,\quad -Cc_{24}^2=\lambda^4 u_2 u_5 u_7 u_9,\quad -Cc_{25}^2=\lambda^4 u_3 u_4 u_8 u_9,$$

$$-Cc_{34}^2=\lambda^4 u_3 u_4 u_7 u_{10},\quad -Cc_{35}^2=\lambda^4 u_2 u_5 u_8 u_{10},\quad -Cc_{45}^2=\lambda^4 u_1 u_6 u_9 u_{10}.$$

Also from the sixth and last two equations of (8) we get

$$(c_{16}^2 c_{12} c_{13})/c_{23}=(\lambda u_9 u_{10})/u_6,$$

and so

$$c_{16}^2=\frac{u_9 u_{10}}{u_1 u_6}c_{23}^2.$$

Thus $-Cc_{16}^2=\lambda^4 u_7 u_8 u_9 u_{10}$, and similarly

$$-Cc_{26}^2=\lambda^4 u_4 u_5 u_6 u_{10},\qquad -Cc_{36}^2=\lambda^4 u_2 u_3 u_6 u_9,$$

$$-Cc_{46}^2=\lambda^4 u_1 u_3 u_5 u_8,\qquad -Cc_{56}^2=\lambda^4 u_1 u_2 u_4 u_7.$$

By use of equations (9) to (13) it is easy to show that

$$\begin{aligned}
&u_1u_2u_3u_{10}+u_1u_4u_5u_9+u_2u_4u_6u_8+u_3u_5u_6u_7+u_7u_8u_9u_{10}\\
&=u_1u_2u_3u_{10}+u_1u_6u_7u_8+u_2u_5u_7u_9+u_3u_4u_8u_9+u_4u_5u_6u_{10}\\
&=u_1u_4u_5u_9+u_1u_6u_7u_8+u_3u_4u_7u_{10}+u_2u_5u_8u_{10}+u_2u_3u_6u_9\\
&=u_2u_4u_6u_8+u_2u_5u_7u_9+u_3u_4u_7u_{10}+u_1u_6u_9u_{10}+u_1u_3u_5u_8\\
&=u_3u_5u_6u_7+u_3u_4u_8u_9+u_2u_5u_8u_{10}+u_1u_6u_9u_{10}+u_1u_2u_4u_7\\
&=u_7u_8u_9u_{10}+u_4u_5u_6u_{10}+u_2u_3u_6u_9+u_1u_3u_5u_8+u_1u_2u_4u_7\\
&=u_1^2u_6^2+u_2^2u_5^2+u_3^2u_4^2+2u_2u_3u_4u_5-2u_1u_3u_4u_6+2u_1u_2u_5u_6\\
&=V.
\end{aligned}$$

Thus, since c_{ij} is orthogonal, we get $c_{12}^2=(u_1u_2u_3u_{10})/V$, etc.*

The alternative signs for c_{ij} can now be chosen in any way so long as equations (9) to (13) are satisfied after taking products of pairs of row elements.

* The number λ is, in fact, $(u_1 u_2 \dots u_{10})^{\frac{1}{4}} V^{-\frac{1}{2}}$.

One such choice gives the orthogonal matrix to be

$$\begin{bmatrix} 0, & (12310)', & (1459)', & (2468)', & (3567)', & (78910)' \\ (12310)', & 0, & (1678)', & -(2579)', & -(3489)', & (45610)' \\ (1459)', & (1678)', & 0, & -(34710)', & (25810)', & -(2369)' \\ (2468)', & -(2579)', & -(34710)', & 0, & (16910)', & (1358)' \\ (3567)', & -(3489)', & (25810)', & (16910)', & 0, & -(1247)' \\ (78910)', & (45610)', & -(2369)', & (1358)', & -(1247)', & 0, \end{bmatrix}, \tag{14}$$

where $(pqrs)'$ is written for $(u_p u_q u_r u_s)^{\frac{1}{2}}/V^{\frac{1}{2}}$.

If we take the positive sign with $V^{\frac{1}{2}}$ and so ignore reversal of sign of every element, the complete solution is given by the operations of the group E, G_1, G_2,..., G_6 upon the above matrix, where G_r is the operation of changing the signs of the elements of row r and then of column r. There are thirty-two such operations.

4. The general symmetric zero-axial matrix A of order six in which the six principal diagonal minor determinants are also zero is thus

$$\Gamma \mathrm{B} \Gamma, \tag{15}$$

where Γ is the general diagonal matrix with diagonal γ_1, γ_2,..., γ_6 and B is (14) with the accents removed, where now $(pqrs)$ is written for $(u_p u_q u_r u_s)^{\frac{1}{2}}$. Thus A contains ten arbitrary parameters.

So, finally, the general quadric primal in [5] which is at the same time inscribed and circumscribed to the simplex of reference is the quadric the matrix of whose coefficients is (15). Its equation contains nine arbitrary parameters. If the parameters u are so chosen that $V = 0$, then the quadric degenerates into a cone with plane vertex since the matrix (15) is then of rank three. The ∞^8 vertex planes are, in fact, the planes which lie upon all quadrics with respect to which the given simplex is self-conjugate.*

By taking one such quadric for which, say, Γ is the unit matrix we obtain the well-known representation of an involution of sixteen points or planes of [3] by the points of the Segre primal V, the involution being given by the alternative signs in the matrix B.

* The equivalent geometric property is that the six solids through a plane upon a quadric primal and the six vertices in turn of a self-conjugate simplex lie upon a quadric plane vertex cone, and this is equivalent to the familiar property that the six poles of a given plane with respect to six linear complexes in involution lie upon a conic.

ON TARRY'S PROBLEM

By LOO-KENG HUA (*Kunming, China*)

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LET $M(k)$ be the least value of s such that the equations

$$a_1^h + \ldots + a_s^h = b_1^h + \ldots + b_s^h \quad (1 \leqslant h \leqslant k) \tag{1}$$

and

$$a_1^{k+1} + \ldots + a_s^{k+1} \neq b_1^{k+1} + \ldots + b_s^{k+1} \tag{2}$$

have a solution in integers. In this paper I shall prove that

$$M(k) \leqslant (k+1)\left(\left[\frac{\log \frac{1}{2}(k+2)}{\log(1+1/k)}\right]+1\right).$$

The method used is very elementary and no previous knowledge is assumed.*

Throughout the paper, $c_1, c_2, \ldots$ denote positive numbers depending on k only.

LEMMA 1. *Given any positive H, there exists a set of positive integers $a_1, \ldots, a_k$ (depending only on k, H) such that the product of the principal diagonal of the determinant*

$$D_k = \begin{vmatrix} 1 & . & . & . & . & 1 \\ a_1 & . & . & . & . & a_k \\ & . & . & . & . & \\ a_1^{k-1} & & . & . & . & a_k^{k-1} \end{vmatrix}$$

is greater than H times the sum of the absolute values of all the other terms in the expansion of the determinant.

Proof. We prove the lemma by induction. If $\phi_j(a_1, \ldots, a_j)$ denotes the product of the principal diagonal minus H times the sum of the absolute values of the other terms in the determinant D_j $(j \leqslant k)$, then

$$\phi_j(a_1, \ldots, a_j) = a_j^{j-1}\phi_{j-1}(a_1, \ldots, a_{j-1}) - H\psi(a_1, \ldots, a_j),$$

where ψ is a polynomial of degree $j-2$ in a_j. Thus, if $a_1, \ldots, a_{j-1}$ have been chosen to make ϕ_{j-1} positive, we can further choose a_j so large that ϕ_j is positive. But initially $\phi_1 = 1$. Thus the induction is established.

* By using a lemma of I. Vinogradoff's we can obtain clearer information about the number of solutions of (1) and (2) with $0 \leqslant a_i \leqslant P$, $0 \leqslant b_i \leqslant P$. More precisely, by that method we are able to prove that the number $r(P)$ of these solutions satisfies

$$cP^{2s-\frac{1}{2}k(k+1)} \leqslant r(P) \leqslant c'P^{2s-\frac{1}{2}k(k+1)},$$

where c and c' are two numbers depending only on k.

LEMMA 2. *Let $X_1,\ldots, X_k$ be integers lying in the intervals*

$$a_i Q \leqslant X_i \leqslant 2a_i Q,$$

where the a's are those defined in Lemma 1. Let N be the number of sets $(X_1,\ldots, X_k)$ for which

$$X_1^k+\ldots+X_k^k,\quad X_1^{k-1}+\ldots+X_k^{k-1},\quad \ldots,\quad X_1+\ldots+X_k$$

lie in given intervals of lengths

$$O(Q^{k-1}),\quad O(Q^{k-2}),\quad \ldots,\quad O(Q),\quad O(1)$$

respectively. Then

$$N = O(1).$$

Proof. If $(X_1,\ldots, X_k)$ and $(X_1',\ldots, X_k')$ are two sets satisfying the requirements of the lemma, then

$$X_1^k-X_1'^k+\ldots+X_k^k-X_k'^k = O(Q^{k-1}),$$

$$\cdots\cdots\cdots\cdots$$

$$X_1-X_1'+\ldots+X_k-X_k' = O(1).$$

Let $Y_i = X_i-X_i'$. Then we have

$$A_{11}Y_1+\ldots+A_{1k}Y_k = O(Q^{k-1}),$$

$$\cdots\cdots\cdots\cdots$$

$$A_{k1}Y_1+\ldots+A_{hk}Y_k = O(1),$$

where
$$A_{ij} = X_j^{k-i}+X_j^{k-i-1}X_j'+\ldots+X_j'^{h-i}.$$

Then
$$(k-i+1)(a_j Q)^{k-i} \leqslant A_{ij} \leqslant (k-i+1)(2a_j Q)^{k-i}.$$

Consider the determinant $|A_{ij}|$. The product of the elements in the principal diagonal divided by the corresponding term of D_k is greater than

$$k!\, Q^{k-1+k-2+\ldots+2+1} = k!\, Q^{\frac{1}{2}k(k-1)}.$$

Further, the absolute values of the other terms in the expansion of $|A_{ij}|$ are less than

$$2^{\frac{1}{2}k(k-1)}k!\, Q^{\frac{1}{2}k(k-1)}$$

times the absolute values of the corresponding terms in D_k. By Lemma 1, with $H = 2^{\frac{1}{2}k(k-1)}$, we have

$$|A_{ij}| \geqslant c_1\, Q^{\frac{1}{2}k(k-1)}.$$

Further,

$$\begin{vmatrix} O(Q^{k-1}) & a_{12} & \cdot & \cdot & \cdot & a_{1k} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ O(1) & a_{k2} & \cdot & \cdot & \cdot & a_{kk} \end{vmatrix} = O(Q^{\frac{1}{2}k(k-1)}).$$

Thus
$$Y_1 = O(1).$$

Similarly, $\qquad Y_2 = O(1), \quad ..., \quad Y_k = O(1).$

Thus we have the lemma.

Let R_k be the number of solutions of

$$\sum_{j=1}^{n}\sum_{i=1}^{k} \chi_{ij}^{h} = \sum_{j=1}^{n}\sum_{i=1}^{k} \chi_{ij}'^{h} \quad (1 \leqslant h \leqslant k) \tag{3}$$

where the χ satisfy the conditions

$$a_i P^{(1-1/k)^{j-1}} \leqslant \chi_{ij} \leqslant 2a_i P^{(1-1/k)} \quad (i = 1, 2, ..., k), \tag{4}$$

and the χ' satisfy the same conditions. Let R_k' be the number of solutions of the equations

$$\sum_{j=1}^{n}\sum_{i=1}^{k} \chi_{ij}^{h} = \sum_{j=1}^{n}\sum_{i=1}^{k} \chi_{ij}'^{h} \quad (1 \leqslant h \leqslant k-1),$$

where the χ and χ' satisfy the same conditions.

LEMMA 3. $\qquad R_k' \geqslant c_2 P^{2k^2\{1-(1-1/k)^n\}-\frac{1}{2}k(k-1)}.$

Proof. Let $r(n_1,..., n_{k-1})$ be the number of solutions of the equations

$$\sum_{j=1}^{n}\sum_{i=1}^{k} \chi_{ij}^{h} = n_h \quad (1 \leqslant h \leqslant k-1)$$

satisfying the conditions (4). Then, evidently,

$$\sum_{n_1} \cdots \sum_{n_{k-1}} r(n_1,..., n_{k-1}) \geqslant c_3 P^{k\{1+(1-1/k)+...+(1-1/k)^{n-1}\}}$$
$$= c_3 P^{k^2\{1-(1-1/k)^n\}}$$

where the summation runs over all possible sets $n_1,..., n_k$. By Schwarz's inequality we have

$$\sum_{n_1} \cdots \sum_{n_{k-1}} r(n_1,..., n_{k-1}) \leqslant \sqrt{\Big\{\sum_{n_1} \cdots \sum_{n_{k-1}} 1 \sum_{n_1} \cdots \sum_{n_{k-1}} r^2(n_1,..., n_{k-1})\Big\}}$$
$$\leqslant \sqrt{\Big\{c_4 P^{1+2+...+k-1} \sum_{n_1} \cdots \sum_{n_{k-1}} r^2(n_1,..., n_{k-1})\Big\}},$$

since $c_5 P^h \leqslant n_h \leqslant c_6 P^h$. Thus

$$R_k' = \sum_{n_1} \cdots \sum_{n_{k-1}} r^2(n_1,..., n_{k-1})$$
$$\geqslant c_2 P^{2k^2\{1-(1-1/k)^n\}-\frac{1}{2}k(k-1)}.$$

LEMMA 4. $\qquad R_k = O(P^{\{2k^2-\frac{1}{2}k(k+1)^n\}\{1-(1-1/k)^n\}}).$

Proof. From (3) and (4) we have

$$\sum_{i=1}^{k} \chi_{i1}^{h} - \sum_{i=1}^{k} \chi_{i1}'^{k} = O(P^{h(1-1/k)}) \quad (1 \leqslant h \leqslant k).$$

Then, for fixed χ_{i1}' $(i = 1,..., k)$,

$$\sum_{i=1}^{k} \chi_{i1}^{k}, \quad \sum_{i=1}^{k} \chi_{i1}^{k-1}, \quad ..., \quad \sum_{i=1}^{k} x_{i1}$$

lie in intervals of the lengths

$$O(P^{k(1-1/k)}),\quad O(P^{(k-1)(1-1/k)}),\quad \dots,\quad O(P^{(1-1/k)}) \tag{5}$$

respectively. Since the system of intervals (5) can be divided into

$$O\left(\frac{P^{k(1-1/k)}}{P^{k-1}}\frac{P^{(k-1)(1-1/k)}}{P^{k-2}}\cdots\frac{P^{2(1-1/k)}}{P}\frac{P^{1-1/k}}{1}\right) = O(P^{k-\frac{1}{2}(k+1)})$$

systems of intervals of lengths

$$O(P^{k-1}),\quad O(P^{k-2}),\quad \dots,\quad O(P),\quad O(1),$$

by Lemma 2 (with $Q = P$), the number of systems of χ_{i1} $(i = 1,\dots,k)$ is

$$O(P^{k-\frac{1}{2}(k+1)}).$$

Therefore the number of systems of χ_{i1} and χ'_{i1} $(i = 1,\dots,k)$ is

$$O(P^{2k-\frac{1}{2}(k+1)}).$$

Further, for fixed χ_{ij}, χ'_{ij} $(1 \leqslant i \leqslant k;\ 1 \leqslant j \leqslant l-1)$ and $\chi'_{i,l}$ $(1 \leqslant i \leqslant k)$, we see by (3) and (4) that

$$\sum_{i=1}^{k}\chi_{il}^{k},\quad \sum_{i=1}^{k}\chi_{il}^{k-1},\quad \dots,\quad \sum_{i=1}^{k}\chi_{il}$$

lie in intervals of the lengths

$$O(P^{k(1-1/k)^l}),\quad O(P^{(k-1)(1-1/k)^l}),\quad \dots,\quad O(P^{(1-1/k)^l}) \tag{6}$$

respectively. Since

$$O\left(\frac{P^{k(1-1/k)^l}}{P^{(k-1)(1-1/k)^{l-1}}}\cdot\frac{P^{(k-1)(1-1/k)^l}}{P^{(k-2)(1-1/k)^{l-1}}}\cdots\frac{P^{(1-1/k)^l}}{1}\right) = O(P^{(k-\frac{1}{2}(k+1))(1-1/k)^{l-1}}),$$

by Lemma 2 (with $Q = P^{(1-1/k)^{l-1}}$), the number of systems $\chi_{i,l}$ $(1 \leqslant i \leqslant k)$ is

$$O(P^{\{k-\frac{1}{2}(k+1)\}(1-1/k)^{l-1}}).$$

Therefore, for fixed χ_{ij}, χ'_{ij} $(1 \leqslant i \leqslant k;\ 1 \leqslant j \leqslant l-1)$, the number of systems of χ_{il} and χ'_{il}, is

$$O(P^{\{2k-\frac{1}{2}(k+1)\}(1-1/k)^{l-1}}).$$

Thus the total number of solutions of (3) with the restriction (4) is

$$O(P^{\{2k-\frac{1}{2}(k+1)\}\{1+(1-1/k)+\dots+(1-1/k)^{n-1}\}})$$
$$= O(P^{\{2k^2-\frac{1}{2}k(k+1)\}\{1-(1-1/k)^n\}}).$$

THEOREM. *If* $n > \log \frac{1}{2}(k+1)/\{\log k - \log(k-1)\}$, *then there are infinitely many sets of integers satisfying*

$$\sum_{j=1}^{n}\sum_{i=1}^{k}\chi_{ij}^{h} = \sum_{j=1}^{n}\sum_{i=1}^{k}\chi_{ij}'^{h} \quad (1 \leqslant h \leqslant k-1)$$

and

$$\sum_{j=1}^{n}\sum_{i=1}^{k}\chi_{ij}^{k} \neq \sum_{j=1}^{n}\sum_{i=1}^{k}\chi_{ij}'^{k}.$$

Proof. We consider those χ_{ij} and χ'_{ij} satisfying (4). The number of solutions of the equations in the theorem is evidently equal to

$$R'_k - R_k.$$

By Lemmas 3 and 4,

$$\begin{aligned} R'_k - R_k &\geqslant c_2 P^{2k^2\{1-(1-1/k)^n\}-\frac{1}{2}k(k-1)} - O\left(P^{\{2k^2-\frac{1}{2}k(k+1)\}\{1-(1-1/k)^n\}}\right) \\ &\geqslant c_7 P^{2k^2\{1-(1-1/k)^n\}-\frac{1}{2}k(k-1)} \end{aligned}$$

for sufficiently large P, since

$$n > \frac{\log \frac{1}{2}(k+1)}{\log k - \log(k-1)}.$$

Hence the theorem follows immediately.

Replacing k by $k+1$ in the theorem, we have consequently

$$M(k) \leqslant (k+1)\left(\left[\frac{\log \frac{1}{2}(k+2)}{\log(1+1/k)}\right]+1\right).$$

This asserts that $$\overline{\lim_{k\to\infty}} \frac{M(k)}{k^2 \log k} \leqslant 1.$$

In what follows I shall indicate a method of improving the constant on the right-hand side, but I am unable to obtain anything better than that

$$M(k) = O(k^2 \log k).$$

Let $k = 2l-1$ be an odd integer. Let $J(l)$ be the least integer s for which

$$\sum_{i=1}^{s} \chi_i^{2h} = \sum_{i=1}^{s} \chi_i'^{2h} \quad (h = 1,\ldots, l-1)$$

and

$$\sum_{i=1}^{s} \chi^{2l} \neq \sum_{i=1}^{s} \chi_i'^{2l}$$

is solvable. By the same method as before, we can show that

$$J(l) \leqslant l\left[\frac{\log \frac{1}{2}(l+1)}{\log l - \log(l-1)}+1\right].$$

Evidently, if

$$\begin{gathered} a_1^{2h}+\ldots+a_s^{2h} = b_1^{2h}+\ldots+b_s^{2h} \quad (1 \leqslant h \leqslant l-1), \\ a_1^{2l}+\ldots+a_s^{2l} \neq b_1^{2l}+\ldots+b_s^{2l}, \end{gathered}$$

then

$$\sum_{i=1}^{s} \{(x+a_i)^t+(x-a_i)^t\} = \sum_{i=1}^{s} \{(x+b_i)^t+(x-b_i)^t\} \quad (1 \leqslant t \leqslant 2l-1)$$

and

$$\sum_{i=1}^{s} \{(x+a_i)^{2l}+(x-a_i)^{2l}\} \neq \sum_{i=1}^{s} \{(x+b_i)^{2l}+(x-b_i)^{2l}\}.$$

Thus we have

$$M(k) \leqslant 2J(l) \leqslant 2l\left[\frac{\log \frac{1}{2}(l+1)}{\log l - \log(l-1)} + 1\right]$$

$$\leqslant (k+1)\left[\frac{\log \frac{1}{4}(k+3)}{\log(k+1) - \log(k-1)} + 1\right] \sim \tfrac{1}{2}k^2 \log k.$$

The other method is the reconsideration of the 'tails' of (3), i.e. the parts corresponding to χ_{in} and χ'_{in}. Such a method will sharpen the result by subtracting a number ($> \frac{1}{2}k$).

Remarks. 1. For $k \geqslant 15$, the result here given is better than E. M. Wright's result that, if $k \geqslant 12$, then

$$M(k) < \frac{7k^2(k-11)(k+3)}{216}.$$

2. The theorem holds good, if we make the restriction that the a and b are primes.

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C51